

Leonard pairs having zero-diagonal TD-TD form

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Abstract

Fix an algebraically closed field \mathbb{F} and an integer $n \geq 1$. Let $\text{Mat}_n(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $n \times n$ matrices that have all entries in \mathbb{F} . We consider a pair of diagonalizable matrices in $\text{Mat}_n(\mathbb{F})$, each acting in an irreducible tridiagonal fashion on an eigenbasis for the other one. Such a pair is called a Leonard pair in $\text{Mat}_n(\mathbb{F})$. In the present paper, we find all Leonard pairs A, A^* in $\text{Mat}_n(\mathbb{F})$ such that each of A and A^* is irreducible tridiagonal with all diagonal entries 0. This solves a problem given by Paul Terwilliger.

1 Introduction

Throughout the paper \mathbb{F} denotes an algebraically closed field. Fix an integer $d \geq 0$ and a vector space V over \mathbb{F} with dimension $d + 1$. Let \mathbb{F}^{d+1} denote the \mathbb{F} -vector space consisting of the column vectors of length $d + 1$, and $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $(d + 1) \times (d + 1)$ matrices that have all entries in \mathbb{F} . We index rows and columns by $0, 1, \dots, d$. The algebra $\text{Mat}_{d+1}(\mathbb{F})$ acts on \mathbb{F}^{d+1} by left multiplication.

We begin by recalling the notion of a Leonard pair. We use the following terms. A square matrix is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1 (See [19, Definition 1.1].) By a *Leonard pair on V* we mean an ordered pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (i) and (ii) below:

- (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We say A, A^* has *diameter d* . By a *Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$* we mean an ordered pair of matrices A, A^* in $\text{Mat}_{d+1}(\mathbb{F})$ that acts on \mathbb{F}^{d+1} as a Leonard pair.

Note 1.2 According to a common notational convention, A^* denotes the conjugate transpose of A . We are not using this convention. In a Leonard pair A, A^* the matrices A and A^* are arbitrary subject to the conditions (i) and (ii) above.

We refer the reader to [16–23, 25, 26] for background on Leonard pairs. Paul Terwilliger gave the following problems.

Problem 1.3 (See [23, Problem 36.14].) Find all Leonard pairs A, A^* in $\text{Mat}_{d+1}(\mathbb{F})$ that satisfy the following conditions: (i) A is lower bidiagonal with all subdiagonal entries 1; (ii) A^* is irreducible tridiagonal.

Problem 1.4 (See [23, Problem 36.16].) Find all Leonard pairs A, A^* in $\text{Mat}_{d+1}(\mathbb{F})$ such that each of A, A^* is irreducible tridiagonal with all diagonal entries 0.

In [14] we gave a partial solution of Problem 1.3. In the present paper we solve Problem 1.4. To state our main results, we first recall the notion of an isomorphism of Leonard pairs. Let A, A^* be a Leonard pair on V and let B, B^* be a Leonard pair on a vector space V' with dimension $d+1$. By an *isomorphism of Leonard pairs* from A, A^* to B, B^* we mean a linear bijection $\sigma : V \rightarrow V'$ such that both $\sigma A = B\sigma$ and $\sigma A^* = B^*\sigma$. We say two Leonard pairs A, A^* and B, B^* are *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to B, B^* . We use the following term:

Definition 1.5 A matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$ is said to be *zero-diagonal TD* whenever A is irreducible tridiagonal with all diagonal entries 0. A pair of matrices A, A^* in $\text{Mat}_{d+1}(\mathbb{F})$ is said to be *zero-diagonal TD-TD* whenever each of A, A^* is zero-diagonal TD.

Note 1.6 The following hold for nonzero $\xi, \xi^* \in \mathbb{F}$.

- (i) Let A, A^* be a Leonard pair on V . Then $\xi A, \xi^* A^*$ is a Leonard pair on V .
- (ii) Let A, A^* be a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$. Then $\xi B, \xi^* B$ is a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$.

Definition 1.7 Let A, A^* be a Leonard pair on V . By the *opposite* of A, A^* we mean the Leonard pair $-A, -A^*$.

We are now ready to state our first main result.

Theorem 1.8 Let A, A^* be a Leonard pair on V . Then the following (i) and (ii) are equivalent:

- (i) There exists a basis for V with respect to which the matrices representing A, A^* form a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$.
- (ii) A, A^* is isomorphic to its opposite.

Note 1.9 In Theorem 1.8, the implication (i) \Rightarrow (ii) is immediate from the following observation. Consider the diagonal matrix $D \in \text{Mat}_{d+1}(\mathbb{F})$ that has (i, i) -entry $(-1)^i$ for $0 \leq i \leq d$. Let $A \in \text{Mat}_{d+1}(\mathbb{F})$ be a zero-diagonal TD matrix. Then $D^{-1}AD = -A$.

Theorem 1.8 is related to a class of Leonard pairs, called totally bipartite. It is known that a totally bipartite Leonard pair is isomorphic to its opposite. (see [24, Chapter 2, Lemma 38]). See [1, 5, 13, 24] for more information concerning totally bipartite Leonard pairs.

Below we describe the parameter array of a Leonard pair that is isomorphic to its opposite (see Definition 2.6 for the definition of a parameter array).

Proposition 1.10 *Let A, A^* be a Leonard pair on V with parameter array*

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d).$$

Then the following (i) and (ii) are equivalent:

- (i) A, A^* is isomorphic to its opposite.
- (ii) The parameter array satisfies

$$\begin{aligned} \theta_i + \theta_{d-i} &= 0, & \theta_i^* + \theta_{d-i}^* &= 0 & (0 \leq i \leq d), \\ \varphi_i &= \varphi_{d-i+1}, & \phi_i &= \phi_{d-i+1} & (1 \leq i \leq d). \end{aligned}$$

We handle the case $d \leq 2$ in Section 4. For the rest of this section, assume $d \geq 3$. In this case, the fundamental parameter β is well-defined (see Definition 2.12 for the definition). In [22] Terwilliger gave a classification of Leonard pairs. By that classification, Leonard pairs are classified into thirteen types. For a Leonard pair that is isomorphic to its opposite, the type is as follows (see Definition 6.8 for the definition of these types).

Proposition 1.11 *Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Let β be the fundamental parameter of A, A^* .*

- (i) *Assume $\beta = 2$. Then A, A^* has Krawtchouk type.*
- (ii) *Assume $\beta = -2$. Then A, A^* has Bannai-Ito type with even diameter.*
- (iii) *Assume $\beta \neq 2$ and $\beta \neq -2$. Then A, A^* has q -Racah type.*

In Section 7 we display five families of zero-diagonal TD-TD Leonard pairs in $\text{Mat}_{d+1}(\mathbb{F})$. See Propositions 7.1–7.5. Among these five families, the family in Proposition 7.3 is the most general one. This family comes from the “compact basis” given by Ito-Rosengren-Terwilliger (see [8, Section 17]). The compact basis is obtained from an evaluation module for the q -tetrahedron algebra. See [3, 8, 9, 12] about the q -tetrahedron algebra. The families in Propositions 7.1, 7.2, 7.4 are related to “Leonard triples”. See [1, 2, 4–7, 11] about Leonard triples. The family in Proposition 7.5 is somewhat mysterious, and the author has no conceptual explanation for this family.

Proposition 1.12 *Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Let β be the fundamental parameter of A, A^* . Then after replacing A, A^* with their nonzero scalar multiples if necessary, the following hold.*

- (i) *Assume $\beta = 2$. Then A, A^* is represented by a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$ that belongs to the family in Proposition 7.1.*
- (ii) *Assume $\beta = -2$. Then A, A^* is represented by a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$ that belongs to the family in Proposition 7.2.*
- (iii) *Assume $\beta \neq 2$ and $\beta \neq -2$. Then A, A^* is represented by a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$ that belongs to the family in Proposition 7.3.*

Theorem 1.8(ii) \Rightarrow (i) immediately follows from Proposition 1.12. To state our second main result, we make some observations and definitions.

Note 1.13 Let A, A^* be a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$, and let $D \in \text{Mat}_{d+1}(\mathbb{F})$ be an invertible diagonal matrix. Then $D^{-1}AD, D^{-1}A^*D$ is a zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, if A, A^* is a Leonard pair, then $D^{-1}AD, D^{-1}A^*D$ is a Leonard pair that is isomorphic to A, A^* .

Definition 1.14 Let A, A^* and B, B^* be zero-diagonal TD-TD pairs in $\text{Mat}_{d+1}(\mathbb{F})$. We say A, A^* and B, B^* are *equivalent* whenever there exists an invertible diagonal matrix $D \in \text{Mat}_{d+1}(\mathbb{F})$ such that $B = D^{-1}AD$ and $B^* = D^{-1}A^*D$.

Note 1.15 Let $A \in \text{Mat}_{d+1}(\mathbb{F})$ be a zero-diagonal TD-TD matrix that has subdiagonal entries $\{x_i\}_{i=1}^d$. Let $D \in \text{Mat}_{d+1}(\mathbb{F})$ be the diagonal matrix that has (i, i) -entry $x_1 x_2 \cdots x_i$ for $0 \leq i \leq d$. Then $D^{-1}AD$ is a zero-diagonal TD-TD matrix that has all subdiagonal entries 1.

Note 1.16 Let $A \in \text{Mat}_{d+1}(\mathbb{F})$ be a zero-diagonal TD matrix with subdiagonal entries $\{x_i\}_{i=1}^d$ and superdiagonal entries $\{y_i\}_{i=1}^d$. Then the anti-diagonal transpose of A has subdiagonal entries $\{x_{d-i+1}\}_{i=1}^d$ and superdiagonal entries $\{y_{d-i+1}\}_{i=1}^d$. Observe that the anti-diagonal transpose of A is $Z^{-1}A^T Z$, where $Z \in \text{Mat}_{d+1}(\mathbb{F})$ has (i, j) -entry $\delta_{i, d-j}$ for $0 \leq i, j \leq d$, and A^T denotes the transpose of A . Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. By [23, Theorem 2.2] the anti-diagonal transpose of A and A^* form a Leonard pair that is isomorphic to A, A^* . We call this Leonard pair the *anti-diagonal transpose* of A, A^* .

We are now ready to state our second main result:

Theorem 1.17 *Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with fundamental parameter β .*

- (i) *Assume $\beta = 2$. Then, after replacing A, A^* with their nonzero scalar multiples if necessary, A, A^* is equivalent to a zero-diagonal TD-TD pair that belongs to the family in Proposition 7.1.*
- (ii) *Assume $\beta = -2$. Then, after replacing A, A^* with their nonzero scalar multiples if necessary, A, A^* is equivalent to a zero-diagonal TD-TD pair that belongs to the family in Proposition 7.2.*
- (iii) *Assume $\beta \neq 2$ and $\beta \neq -2$. Then, after replacing A, A^* with their nonzero scalar multiples if necessary, A, A^* or its anti-diagonal transpose is equivalent to a zero-diagonal TD-TD pair that belongs to one of the families in Propositions 7.3–7.5.*

The paper is organized as follows. In Section 2 we recall some materials concerning Leonard pairs. In Section 3 we prove Proposition 1.10. In Section 4 we handle the case $d \leq 2$. In Sections 5–20 we assume $d \geq 3$. In Section 5 we recall some formulas that represent the parameter array in closed form. In Section 6 we display formulas for the parameter array of a Leonard pair that is isomorphic to its opposite. Using these formulas we prove Proposition 1.11. In Section 7 we display five families of zero-diagonal TD-TD Leonard pairs in $\text{Mat}_{d+1}(\mathbb{F})$. In Section 8 we recall the Askey-Wilson relations for a Leonard pair. In Section 9 we display a formula for the characteristic polynomial of a zero-diagonal TD matrix in $\text{Mat}_{d+1}(\mathbb{F})$. In Sections 10–14 we prove Propositions 7.1–7.5. In Section 15 we prove Proposition 1.12. In Section 16 we evaluate the Askey-Wilson relations for a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$, and obtain some relations between the entries of the matrices. In Section 17 we obtain some equations for later use. In Sections 18–20 we prove Theorem 1.17.

2 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra.

Let $A : V \rightarrow V$ be a linear transformation. We say A is *multiplicity-free* whenever it has $d + 1$ mutually distinct eigenvalues in \mathbb{F} . Assume A is multiplicity-free, and let $\{\theta_i\}_{i=0}^d$ be the eigenvalues of A . For $0 \leq i \leq d$ define

$$E_i = \prod_{\substack{0 \leq \ell \leq d \\ \ell \neq i}} \frac{A - \theta_\ell I}{\theta_i - \theta_\ell}.$$

Here I denotes the identity. Observe (i) $AE_i = \theta_i E_i$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (iii) $I = \sum_{i=0}^d E_i$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Also observe $V = \sum_{i=0}^d E_i V$ (direct sum), and E_i acts on V as the projection onto $E_i V$. We call E_i the *primitive idempotent* of A associated with θ_i . We now define a Leonard system.

Definition 2.1 [19] By a *Leonard system* on V we mean a sequence

$$(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free linear transformation from V to V .
- (ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* .
- (iv) $E_i A^* E_j = 0$ and $E_i^* A E_j^* = 0$ if $|i - j| > 1$ for $0 \leq i, j \leq d$.
- (v) $E_i A^* E_j \neq 0$ and $E_i^* A E_j^* \neq 0$ if $|i - j| = 1$ for $0 \leq i, j \leq d$.

Leonard systems are related to Leonard pairs as follows. Let $(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system on V . Then A, A^* is a Leonard pair on V . Conversely, let A, A^* be a Leonard pair on V . Then each of A, A^* is multiplicity-free (see [19, Lemma 1.3]). Moreover there exists an ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents of A , and there exists an ordering $\{E_i^*\}_{i=0}^d$ of the primitive idempotents of A^* , such that $(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ is a Leonard system on V . We say the Leonard pair A, A^* and the Leonard system $(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ are *associated*.

Definition 2.2 Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system on V . For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) be the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ .

We recall the notion of an isomorphism of Leonard systems. Consider a Leonard system $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ on V and a Leonard system $\Phi' = (A', \{E'_i\}_{i=0}^d, A'^*, \{E'^*_i\}_{i=0}^d)$ on a vector space V' with dimension $d + 1$. By an *isomorphism of Leonard systems from Φ to Φ'* we mean a linear bijection $\sigma : V \rightarrow V'$ such that $\sigma A = A' \sigma$, $\sigma A^* = A'^* \sigma$, and $\sigma E_i = E'_i \sigma$, $\sigma E_i^* = E'^*_i \sigma$ for $0 \leq i \leq d$. Leonard systems Φ and Φ' are said to be *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ' .

Let A, A^* be a Leonard pair on V and let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system associated with A, A^* . Then A, A^* is associated with the following Leonard systems, and no further Leonard systems:

$$\begin{aligned} \Phi^\downarrow &:= (A, \{E_i\}_{i=0}^d, A^*, \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\uparrow &:= (A, \{E_{d-i}\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d), \\ \Phi^{\downarrow\uparrow} &:= (A, \{E_{d-i}\}_{i=0}^d, A^*, \{E_{d-i}^*\}_{i=0}^d). \end{aligned}$$

Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$.

Definition 2.3 (See [23, Section 5.1].) Pick a nonzero $v \in E_0^*V$. For $0 \leq i \leq d$ define

$$u_i = (A - \theta_{i-1}I) \cdots (A - \theta_1I)(A - \theta_0I)v.$$

Then $\{u_i\}_{i=0}^d$ is a basis for V . We call $\{u_i\}_{i=0}^d$ a Φ -split basis for V .

Lemma 2.4 (See [19, Theorem 3.2].) Let $\{u_i\}_{i=0}^d$ be a Φ -split basis for V . Then the matrices representing A, A^* with respect to $\{u_i\}_{i=0}^d$ are

$$A: \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}, \quad (1)$$

for some scalars $\{\varphi_i\}_{i=1}^d$. The sequence $\{\varphi_i\}_{i=0}^d$ is uniquely determined. Moreover $\varphi_i \neq 0$ for $1 \leq i \leq d$.

Definition 2.5 With reference to Lemma 2.4, we call $\{\varphi_i\}_{i=1}^d$ the *first split sequence* of Φ . By the *second split sequence* of Φ we mean the first split sequence of Φ^\downarrow .

Definition 2.6 (See [22, Section 2].) Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system. By the *parameter array* of Φ we mean the sequence

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d), \quad (2)$$

where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ , and $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is the first split sequence (resp. second split sequence) of Φ .

Definition 2.7 Let A, A^* be a Leonard pair on V . By a *parameter array* of A, A^* we mean the parameter array of a Leonard system associated with A, A^* .

Lemma 2.8 (See [15, Theorem 4.6].) Let $(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$. Then for $1 \leq i \leq d$

$$\varphi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^* \prod_{\ell=0}^{i-1} (A - \theta_\ell I))}{\text{tr}(E_0^* \prod_{\ell=0}^{i-2} (A - \theta_\ell I))}, \quad (3)$$

$$\phi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^* \prod_{\ell=0}^{i-1} (A - \theta_{d-\ell} I))}{\text{tr}(E_0^* \prod_{\ell=0}^{i-2} (A - \theta_{d-\ell} I))}. \quad (4)$$

The following two results are fundamental in the theory of Leonard pairs.

Lemma 2.9 (See [19, Theorem 1.9].) A Leonard system is determined up to isomorphism by its parameter array.

Lemma 2.10 (See [19, Theorem 1.9].) *Consider a sequence (2) consisting of scalars taken from \mathbb{F} . Then there exists a Leonard system Φ on V with parameter array (2) if and only if (i)–(v) hold below:*

$$(i) \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad (0 \leq i < j \leq d).$$

$$(ii) \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d).$$

$$(iii) \quad \varphi_i = \phi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).$$

$$(iv) \quad \phi_i = \varphi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$$

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \quad (5)$$

are equal and independent of i for $2 \leq i \leq d-1$.

Definition 2.11 By a *parameter array over \mathbb{F}* we mean a sequence (2) consisting of scalars taken from \mathbb{F} that satisfy conditions (i)–(v) in Lemma 2.10.

Definition 2.12 Assume $d \geq 3$, and let Φ be a Leonard system on V with parameter array (2). Let β be one less than the common value of (5). We call β the *fundamental parameter* of Φ . Let A, A^* be a Leonard pair on V . By the *fundamental parameter* of A, A^* we mean the fundamental parameter of an associated Leonard system.

Lemma 2.13 (See [19, Theorem 1.11].) *Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system with parameter array (2). Then the parameter array of $\Phi^\downarrow, \Phi^\uparrow, \Phi^{\downarrow\downarrow}$ are as follows:*

| Leonard system | Parameter array |
|-------------------------------|---|
| Φ | $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ |
| Φ^\downarrow | $(\{\theta_i\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d, \{\varphi_{d-i+1}\}_{i=1}^d)$ |
| Φ^\uparrow | $(\{\theta_{d-i}\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d, \{\varphi_i\}_{i=1}^d)$ |
| $\Phi^{\downarrow\downarrow}$ | $(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\varphi_{d-i+1}\}_{i=1}^d, \{\phi_{d-i+1}\}_{i=1}^d)$ |

We recall the scalars $\{a_i\}_{i=0}^d$ and $\{a_i^*\}_{i=0}^d$.

Definition 2.14 (See [23, Definition 2.3].) Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system on V . Define scalars $\{a_i\}_{i=0}^d$ and $\{a_i^*\}_{i=0}^d$ by

$$\begin{aligned} a_i &= \text{tr}(E_i^* A) & (0 \leq i \leq d), \\ a_i^* &= \text{tr}(E_i A^*) & (0 \leq i \leq d). \end{aligned}$$

Lemma 2.15 (See [23, Lemma 2.8].) *With reference to Definition 2.14, for $0 \leq i \leq d$ pick a nonzero $v_i \in E_i^*V$. Then $\{v_i\}_{i=0}^d$ be a basis for V . With respect to this basis, the matrix representing A is irreducible tridiagonal with diagonal entries $\{a_i\}_{i=0}^d$, and the matrix representing A^* is diagonal with diagonal entries $\{\theta_i^*\}_{i=0}^d$, where $\{\theta_i^*\}_{i=0}^d$ is the dual eigenvalue sequence of Φ .*

Lemma 2.16 (See [23, Theorem 5.7].) *With reference to Definition 2.14, let (2) be the parameter array of Φ . Then*

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (0 \leq i \leq d), \quad (6)$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \quad (0 \leq i \leq d), \quad (7)$$

where $\varphi_0 = 0$, $\varphi_{d+1} = 0$, and θ_{-1} , θ_{d+1} , θ_{-1}^* , θ_{d+1}^* denote indeterminates.

We recall a scalar multiple of a Leonard system.

Lemma 2.17 (See [17, Lemma 6.1].) *Let $(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system with parameter array (2). Let ξ, ξ^* be nonzero scalars in \mathbb{F} . Then*

$$(\xi A, \{E_i\}_{i=0}^d, \xi^* A^*, \{E_i^*\}_{i=0}^d)$$

is a Leonard system with parameter array

$$(\{\xi\theta_i\}_{i=0}^d, \{\xi^*\theta_i^*\}_{i=0}^d, \{\xi\xi^*\varphi_i\}_{i=1}^d, \{\xi\xi^*\phi_i\}_{i=1}^d).$$

In Definition 2.1 the condition (v) can be slightly weakened as follows. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the linear transformations from V to V .

Lemma 2.18 *Consider a sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ that satisfies conditions (i)–(iv) in Definition 2.1. Then the following (i)–(iii) are equivalent:*

- (i) $E_i A^* E_j \neq 0$ if $|i - j| = 1$ ($0 \leq i, j \leq d$).
- (ii) $E_i^* A E_j^* \neq 0$ if $|i - j| = 1$ ($0 \leq i, j \leq d$).
- (iii) A and A^* together generate $\text{End}(V)$.

Suppose (i)–(iii) hold above. Then Φ is a Leonard system.

Proof. The last assertion is clear. We show (ii) \Leftrightarrow (iii). The proof of (i) \Leftrightarrow (iii) is similar.

(ii) \Rightarrow (iii): For $0 \leq i \leq d$ pick a nonzero $v_i \in E_i^*V$, and note that $\{v_i\}_{i=0}^d$ is a basis for V . We identify each linear transformation with the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents it with respect to $\{v_i\}_{i=0}^d$. Adopting this point of view, A is irreducible tridiagonal and A^* is

diagonal. Moreover, E_0^* has $(0,0)$ -entry 1 and all other entries 0. Using these comments, one finds that

$$(A^r E_0^* A^s) = \begin{cases} 0 & \text{if } i > r \text{ or } j > s, \\ \neq 0 & \text{if } i = r \text{ and } j = s \end{cases} \quad (0 \leq i, j \leq d).$$

Therefore the elements $\{A^r E_0^* A^s \mid 0 \leq r, s \leq d\}$ are linearly independent, and so form a basis for $\text{Mat}_{d+1}(\mathbb{F})$. Observe that E_0^* is a polynomial in A^* by the definition. By these comments A, A^* together generate $\text{Mat}_{d+1}(\mathbb{F})$.

(iii) \Rightarrow (ii): By way of contradiction, assume $E_r^* A E_{r-1}^* = 0$ or $E_{r-1}^* A E_r^* = 0$ for some r ($1 \leq r \leq d$). First assume $E_r^* A E_{r-1}^* = 0$. Then $E_k^* A E_\ell^* = 0$ for $0 \leq \ell < r \leq k \leq d$ by condition (iv) in Definition 2.1. Set $W = \sum_{\ell=0}^{r-1} E_\ell^* V$, and note that $0 \neq W \neq V$. We claim W is invariant under each of A, A^* . Clearly W is invariant under A^* . Using the above comment, we argue $AW = A \sum_{\ell=0}^{r-1} E_\ell^* V = IA \sum_{\ell=0}^{r-1} E_\ell^* V \subseteq \sum_{k=0}^d \sum_{\ell=0}^{r-1} E_k^* A E_\ell^* V = \sum_{k=0}^{r-1} \sum_{\ell=0}^{r-1} E_k^* A E_\ell^* V \subseteq \sum_{k=0}^{r-1} E_k^* V = W$. Therefore W is invariant under A . We have shown the claim. By the assumption, A and A^* generate $\text{End}(V)$, so W is invariant under $\text{End}(V)$. This forces $W = V$, a contradiction. Next assume $E_{r-1}^* A E_r^* = 0$. By considering the subspace $W' = \sum_{\ell=r}^d E_\ell^* V$, we get a contradiction in a similar way as above. \square

3 Some properties of a Leonard pair that is isomorphic to its opposite

In this section we study about the parameter array of a Leonard pair that is isomorphic to its opposite. We then prove Proposition 1.10. The case $d = 0$ is obvious, so we assume $d \geq 1$. Let

$$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$$

be a Leonard system on V with parameter array

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d). \quad (8)$$

Lemma 3.1 *Define*

$$\Phi' = (-A, \{E_i\}_{i=0}^d, -A^*, \{E_i^*\}_{i=0}^d). \quad (9)$$

Then Φ' is a Leonard system with parameter array

$$(\{-\theta_i\}_{i=0}^d, \{-\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d). \quad (10)$$

Proof. Follows from Lemma 2.17. \square

Lemma 3.2 *Assume A, A^* is isomorphic to its opposite. Then*

$$\theta_i + \theta_{d-i} = 0 \quad (0 \leq i \leq d), \quad (11)$$

$$\theta_i^* + \theta_{d-i}^* = 0 \quad (0 \leq i \leq d). \quad (12)$$

Moreover, $\Phi^{\downarrow\downarrow}$ is isomorphic to Φ' , where Φ' is from (9).

Proof. Observe that Φ' is isomorphic to one of $\Phi, \Phi^\downarrow, \Phi^\uparrow, \Phi^{\downarrow\downarrow}$, since $-A, -A^*$ is isomorphic to A, A^* . By this and Lemma 2.13, $\{-\theta_i\}_{i=0}^d$ coincides with $\{\theta_i\}_{i=0}^d$ or $\{\theta_{d-i}\}_{i=0}^d$. If

$\{-\theta_i\}_{i=0}^d$ coincides with $\{\theta_i\}_{i=0}^d$, then $\theta_i = 0$ for $0 \leq i \leq d$, contradicting Lemma 2.10(i). So $\{-\theta_i\}_{i=0}^d$ coincides with $\{\theta_{d-i}\}_{i=0}^d$, and (11) follows. Similarly (12) holds. Therefore Φ' is isomorphic to $\Phi^{\downarrow\downarrow}$. \square

Lemma 3.3 *Assume A, A^* is isomorphic to its opposite. Then*

$$\begin{aligned}\varphi_i &= \varphi_{d-i+1} & (1 \leq i \leq d), \\ \phi_i &= \phi_{d-i+1} & (1 \leq i \leq d).\end{aligned}$$

Proof. By Lemma 3.2 Φ' and $\Phi^{\downarrow\downarrow}$ are isomorphic, so they have the same parameter array. By Lemma 2.13 the parameter array of $\Phi^{\downarrow\downarrow}$ is

$$(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\varphi_{d-i+1}\}_{i=1}^d, \{\phi_{d-i+1}\}_{i=1}^d).$$

Now compare this with (10) to get the results. \square

Lemma 3.4 *Assume A, A^* is isomorphic to its opposite. Then $\text{Char}(\mathbb{F}) \neq 2$.*

Proof. By Lemma 3.2 $\theta_0 = -\theta_d$. If $\text{Char}(\mathbb{F}) = 2$, then $\theta_0 = \theta_d$, contradicting Lemma 2.10(i). \square

Lemma 3.5 *Assume A, A^* is isomorphic to its opposite. Then for $0 \leq i \leq d$*

$$\theta_i = \begin{cases} \neq 0 & \text{if } i \neq d/2, \\ 0 & \text{if } i = d/2, \end{cases} \quad \theta_i^* = \begin{cases} \neq 0 & \text{if } i \neq d/2, \\ 0 & \text{if } i = d/2. \end{cases}$$

Proof. Follows from Lemma 2.10(i) and Lemmas 3.2, 3.4. \square

Proof of Proposition 1.10. (i) \Rightarrow (ii): Follows from Lemmas 3.2 and 3.3.

(ii) \Rightarrow (i). Let Φ' be from (9). We show that $\Phi^{\downarrow\downarrow}$ and Φ' has the same parameter array. By Lemma 2.13 the parameter array of $\Phi^{\downarrow\downarrow}$ is

$$(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\varphi_{d-i+1}\}_{i=1}^d, \{\phi_{d-i+1}\}_{i=1}^d).$$

By Lemma 3.1 the parameter array of Φ' is (10). By condition (ii) in Proposition 1.10, these parameter arrays coincide. By this and Lemma 2.9 $\Phi^{\downarrow\downarrow}$ is isomorphic to Φ' . So A, A^* is isomorphic to $-A, -A^*$. \square

4 The case $d \leq 2$

In this section we consider the case $d \leq 2$. In view of Lemma 3.4 we assume $\text{Char}(\mathbb{F}) \neq 2$. The case $d = 0$ is obvious, so we assume $d = 1$ or $d = 2$. First consider the case $d = 1$.

Proposition 4.1 *For a nonzero $s \in \mathbb{F}$ with $s^2 \neq 1$, the pair*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & s^{-1} \\ s & 0 \end{pmatrix} \quad (13)$$

is a Leonard pair in $\text{Mat}_2(\mathbb{F})$. Moreover, this Leonard pair has parameter array

$$\theta_0 = 1, \theta_1 = -1, \quad \theta_0^* = 1, \theta_1^* = -1, \quad \varphi_1 = s + s^{-1} - 2, \quad \phi_1 = s + s^{-1} + 2. \quad (14)$$

Proof. One routinely checks that the sequence (14) is a parameter array over \mathbb{F} . So there exists a Leonard pair B, B^* that has parameter array (14). By Lemma 2.4 we may assume B, B^* are as in (1):

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & s + s^{-1} - 2 \\ 0 & -1 \end{pmatrix}.$$

Define

$$P = \begin{pmatrix} 1 & s - 1 \\ s & 1 - s \end{pmatrix}.$$

Then $\det P = 1 - s^2 \neq 0$, so P is invertible. One routinely checks that the pair PBP^{-1}, PB^*P^{-1} coincides with the pair (13). So (13) is a Leonard pair that is isomorphic to B, B^* . The result follows. \square

Proposition 4.2 *Assume $d = 1$. Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Then, after replacing A, A^* with their scalar multiples if necessary, there exists a basis for V with respect to which the matrices representing A, A^* are as in Proposition 4.1.*

Proof. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ be a parameter array of A, A^* . Note that $\theta_0 \neq 0$ and $\theta_0^* \neq 0$ by Lemma 3.5. By replacing A, A^* with their scalar multiples, we may assume $\theta_0 = 1$ and $\theta_0^* = 1$. By this and Proposition 1.10, $\theta_d = -1$ and $\theta_d^* = -1$. Pick a nonzero $s \in \mathbb{F}$ such that $\varphi_1 = s + s^{-1} - 2$. By Lemma 2.10(iv) $\phi_1 = s + s^{-1} + 2$. Therefore A, A^* has parameter array as in (14). By this and Proposition 4.1 A, A^* has the same parameter array as the Leonard pair (13). By this and Lemma 2.9 A, A^* is isomorphic to the Leonard pair (13). The result follows. \square

Theorem 1.8(ii) \Rightarrow (i) for $d = 1$ follows from Proposition 4.2.

Proposition 4.3 *Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_2(\mathbb{F})$. Then, after replacing A, A^* with their scalar multiples if necessary, A, A^* is equivalent to the Leonard pair in Proposition 4.1*

Proof. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ be a parameter array of A, A^* . By Theorem 1.8(i) \Rightarrow (ii) A, A^* is isomorphic to its opposite. As in the proof of Proposition 4.2 we may assume $\theta_0 = 1, \theta_1 = -1, \theta_0^* = 1, \theta_1^* = -1$. In view of Note 1.15 we may assume A, A^* take the form:

$$A = \begin{pmatrix} 0 & z_1 \\ 1 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & y_1 z_1 \\ x_1 & 0 \end{pmatrix}$$

for some nonzero scalars $x_1, y_1, z_1 \in \mathbb{F}$. By Lemma 2.4 there exists a basis for \mathbb{F}^2 , with respect to which the matrices representing A, A^* are

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & \varphi_1 \\ 0 & -1 \end{pmatrix}.$$

By the construction, there exists an invertible matrix $P \in \text{Mat}_2(\mathbb{F})$ such that $AP = PB$ and $A^*P = PB^*$. Compute the entries of $AP - PB$ and $A^*P - PB^*$ we obtain some equations. Solving these equations, one finds that $z_1 = 1$ and $y_1 = x_1^{-1}$. Now A, A^* coincides with the pair (13) by setting $s = x_1$. \square

Next consider the case $d = 2$.

Proposition 4.4 *Let $y, z \in \mathbb{F}$ be nonzero scalars such that*

$$y \neq 1, \quad y \neq -1, \quad z \neq 1, \quad yz \neq 1, \quad (y+1)z \neq 2.$$

Then the pair

$$\begin{pmatrix} 0 & z & 0 \\ 1 & 0 & 1-z \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & yz & 0 \\ 1 & 0 & yz-1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (15)$$

is a Leonard pair in $\text{Mat}_3(\mathbb{F})$. Moreover, this Leonard pair has parameter array

$$\begin{aligned} \theta_0 = 1, \theta_1 = 0, \theta_2 = -1, \quad \theta_0^* = 1, \theta_1^* = 0, \theta_2^* = -1, \\ \varphi_1 = \varphi_2 = (y+1)z - 2, \quad \phi_1 = \phi_2 = (y+1)z. \end{aligned} \quad (16)$$

Proof. One routinely checks that the sequence (16) is a parameter array over \mathbb{F} . So there exists a Leonard pair B, B^* that has parameter array (16). By Lemma 2.4 we may assume B, B^* are as in (1):

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & (y+1)z - 2 & 0 \\ 0 & 0 & (y+1)z - 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Define

$$P = \begin{pmatrix} yz & (1-y)z & (y+1)z^2 - 2z \\ 1 & (y+1)z - 2 & 2 - (y+1)z \\ -1 & 2 & (y+1)z - 2 \end{pmatrix}.$$

One checks

$$\det P = (y+1)^2 z^2 ((y+1)z - 2),$$

so P is invertible. One routinely checks that the pair PBP^{-1} , PB^*P^{-1} coincides with the pair (13). So (13) is a Leonard pair that is isomorphic to B, B^* . The result follows. \square

Proposition 4.5 *Let $s, t, z \in \mathbb{F}$ be nonzero scalars such that*

$$s^2 \neq 1, \quad t^2 \neq 1, \quad s+t \neq 0, \quad z \neq 1.$$

Define

$$\begin{aligned} \tilde{y}_1 &= tz + \frac{1-t^2}{s+t}, \\ \tilde{y}_2 &= -sz + \frac{1+st}{s+t}. \end{aligned}$$

Then the pair

$$\begin{pmatrix} 0 & z & 0 \\ 1 & 0 & 1-z \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \tilde{y}_1 & 0 \\ s & 0 & \tilde{y}_2 \\ 0 & t & 0 \end{pmatrix} \quad (17)$$

is a Leonard pair in $\text{Mat}_3(\mathbb{F})$. Moreover, this Leonard pair has parameter array

$$\begin{aligned} \theta_0 &= 1, \theta_1 = 0, \theta_2 = -1, & \theta_0^* &= 1, \theta_1^* = 0, \theta_2^* = -1, \\ \varphi_1 = \varphi_2 &= \frac{(s-1)(t-1)}{s+t}, & \phi_1 = \phi_2 &= \frac{(s+1)(t+1)}{s+t}. \end{aligned} \quad (18)$$

Proof. One routinely checks that the sequence (18) is a parameter array over \mathbb{F} . So there exists a Leonard pair B, B^* that has parameter array (18). By Lemma 2.4 we may assume B, B^* are as in (1):

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & \frac{(s-1)(t-1)}{s+t} & 0 \\ 0 & 0 & \frac{(s-1)(t-1)}{s+t} \\ 0 & 0 & -1 \end{pmatrix}.$$

Define

$$P = \begin{pmatrix} 1-t^2 + (s+t)tz & t^2 - 1 - (s+t)(t-1)z & (s-1)(t-1)z \\ s+t & (s-1)(t-1) & (s-1)(1-t) \\ (s+t)t & (s+t)(1-t) & (s-1)(t-1) \end{pmatrix}.$$

One checks

$$\det P = (1 - s^2)(t^2 - 1)^2,$$

so P is invertible. One routinely checks that the pair PBP^{-1} , PB^*P^{-1} coincides with the pair (13). So (13) is a Leonard pair that is isomorphic to B, B^* . The result follows. \square

Proposition 4.6 *Assume $d = 2$. Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Then, after replacing A, A^* with their scalar multiples if necessary, there exists a basis for V with respect to which the matrices representing A, A^* are as in Proposition 4.4.*

Proof. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ be a parameter array of A, A^* . Note that $\theta_0 \neq 0$, $\theta_0^* \neq 0$ by Lemma 3.5. By replacing A, A^* with their scalar multiples, we may assume $\theta_0 = 1$ and $\theta_0^* = 1$. By this and Proposition 1.10, $\theta_1 = 0$, $\theta_2 = -1$, $\theta_1^* = 0$, $\theta_2^* = -1$. Pick nonzero $y, z \in \mathbb{F}$ such that $\varphi_1 = (y + 1)z - 2$. By Lemma 2.10(iv) $\phi_1 = (y + 1)z$. Therefore A, A^* has parameter array as in (16). By this and Proposition 4.4 A, A^* has the same parameter array as the Leonard pair (15). By this and Lemma 2.9 A, A^* is isomorphic to the Leonard pair (15). The result follows. \square

Theorem 1.8(ii) \Rightarrow (i) for $d = 2$ follows from Proposition 4.6.

Proposition 4.7 *Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_3(\mathbb{F})$. Then, after replacing A, A^* with their scalar multiples if necessary, A, A^* or its anti-diagonal transpose is equivalent to the Leonard pair (15) or (17).*

Proof. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ be a parameter array of A, A^* . By Theorem 1.8(i) \Rightarrow (ii) A, A^* is isomorphic to its opposite. As in the proof of Proposition 4.6 we may assume $\theta_0 = 1$, $\theta_1 = 0$, $\theta_2 = -1$, $\theta_0^* = 1$, $\theta_1^* = 0$, $\theta_2^* = -1$. By Lemma 2.10(iii), (iv),

$$\varphi_2 = \varphi_1, \quad \phi_1 = \varphi_1 + 2, \quad \phi_2 = \varphi_1 + 2. \quad (19)$$

In view of Note 1.15 we may assume A, A^* take the form:

$$A = \begin{pmatrix} 0 & z_1 & 0 \\ 1 & 0 & z_2 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & y_1 z_1 & 0 \\ x_1 & 0 & y_2 z_2 \\ 0 & x_2 & 0 \end{pmatrix}$$

for some nonzero scalars $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{F}$. By Lemma 2.4 there exists a basis for \mathbb{F}^3 , with respect to which the matrices representing A, A^* are

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & \varphi_1 & 0 \\ 0 & 0 & \varphi_1 \\ 0 & 0 & -1 \end{pmatrix}.$$

By the construction, there exists an invertible matrix $P \in \text{Mat}_3(\mathbb{F})$ such that $AP = PB$ and $A^*P = PB^*$. We compute the entries of $AP = PB$ and $A^*P = PB^*$ as follows. In $A^*P = PB^*$, compute the $(0,0)$ and $(2,0)$ entry to find that

$$P_{1,0} = \frac{P_{0,0}}{y_1 z_1}, \quad P_{2,0} = \frac{P_{0,0}(1 - x_1 y_1 z_1)}{y_1 y_2 z_1 z_2}.$$

Observe that $P_{0,0} \neq 0$; otherwise the 0th column of P is 0, contradicting that P is invertible. By replacing P with $P_{0,0}^{-1}P$, we may assume $P_{0,0} = 1$. So

$$P_{1,0} = \frac{1}{y_1 z_1}, \quad P_{2,0} = \frac{1 - x_1 y_1 z_1}{y_1 y_2 z_1 z_2}.$$

In $AP = PB$, compute the $(0,0)$, $(0,1)$, $(1,1)$, $(2,2)$ entries, and in $A^*P = PB^*$, compute the $(0,1)$, $(0,2)$ entries to find that

$$\begin{aligned} P_{0,1} &= \frac{1}{y_1} - 1, & P_{0,2} &= \frac{\varphi_1}{y_1}, \\ P_{1,1} &= \frac{\varphi_1}{y_1 z_1}, & P_{1,2} &= -\frac{\varphi_1}{y_1 z_1}, \\ P_{2,1} &= -\frac{\varphi_1 + z_1 - y_1 z_1}{y_1 z_1 z_2}, & P_{2,2} &= \frac{\varphi_1}{y_1 z_1}. \end{aligned}$$

By $(1,2)$ -entry of $AP = PB$,

$$\varphi_1(z_1 + z_2 - 1) = 0.$$

By this and $\varphi_1 \neq 0$,

$$z_2 = 1 - z_1. \quad (20)$$

Note that $z_1 \neq 1$; otherwise $z_2 = 0$. By $(2,0)$ -entry of $A^*P = PB^*$,

$$1 - x_1 y_1 z_1 + x_2 y_2 (z_1 - 1) = 0.$$

So

$$y_2 = \frac{x_1 y_1 z_1 - 1}{x_2 (z_1 - 1)}. \quad (21)$$

By $(1,0)$ -entry of $AP = PB$,

$$-\varphi_1 + x_2 - x_2 z_1 + y_1 z_1 - 1 = 0.$$

So

$$\varphi_1 = x_2 - x_2 z_1 + y_1 z_1 - 1. \quad (22)$$

By $(1,1)$ -entry of $A^*P = PB^*$,

$$-y_1 z_1 (x_1 + x_2) + x_2^2 (z_1 - 1) + x_1 x_2 z_1 + 1 = 0. \quad (23)$$

First assume $x_1 + x_2 = 0$. Then (23) becomes $x_1^2 = 1$. So either $x_1 = 1$ or $x_1 = -1$. If $x_1 = 1$, then $y_2 z_2 = y_1 z_1 - 1$, and so A, A^* coincides with (15) with $y = y_1$ and $z = z_1$. If $x_1 = -1$, then $y_2 z_2 = y_1 z_1 + 1$, and so

$$A = \begin{pmatrix} 0 & z_1 & 0 \\ 1 & 0 & 1 - z_1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & y_1 z_1 & 0 \\ -1 & 0 & y_1 z_1 + 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Setting

$$z = 1 - z_1, \quad y = \frac{y_1 z_1 + 1}{1 - z_1},$$

the above matrices become

$$A = \begin{pmatrix} 0 & 1 - z & 0 \\ 1 & 0 & z \\ 0 & 1 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & yz - 1 & 0 \\ -1 & 0 & yz \\ 0 & 1 & 0 \end{pmatrix}.$$

This coincides with the anti-diagonal transpose of (15).

Next assume $x_1 + x_2 \neq 0$. By (23)

$$y_1 z_1 = x_2 z_1 + \frac{1 - x_2^2}{x_1 + x_2}.$$

By this and (21)

$$y_2 z_2 = -x_1 z_1 + \frac{1 + x_1 x_2}{x_1 + x_2}.$$

Now A, A^* coincides with the pair (17) by setting $s = x_1$, $t = x_2$, $z = z_1$. The result follows. \square

5 Parameter arrays in closed form

For the rest of the paper we assume $d \geq 3$. In this section we recall the formulas that represent the parameter array in closed form. In view of Lemma 3.4, we assume $\text{Char}(\mathbb{F}) \neq 2$. Let A, A^* be a Leonard pair on V with parameter array

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d),$$

and let β be the fundamental parameter of A, A^* .

Lemma 5.1 (See [17, Lemma 14.1].) *Assume $\beta = 2$. Then there exist scalars $\alpha, h, \mu, \alpha^*, h^*, \mu^*, \tau$ in \mathbb{F} such that*

$$\begin{aligned} \theta_i &= \alpha + \mu(i - d/2) + hi(d - i), \\ \theta_i^* &= \alpha^* + \mu^*(i - d/2) + h^*i(d - i) \end{aligned}$$

for $0 \leq i \leq d$, and

$$\begin{aligned} \varphi_i &= i(d - i + 1)(\tau - \mu\mu^*/2 + (h\mu^* + \mu h^*)(i - (d + 1)/2) + hh^*(i - 1)(d - i)), \\ \phi_i &= i(d - i + 1)(\tau + \mu\mu^*/2 + (h\mu^* - \mu h^*)(i - (d + 1)/2) + hh^*(i - 1)(d - i)) \end{aligned}$$

for $1 \leq i \leq d$.

Note 5.2 (See [17, Remark 14.2].) Referring to Lemma 5.1, $\text{Char}(\mathbb{F})$ is 0 or greater than d .

Lemma 5.3 (See [17, Lemma 15.1].) Assume $\beta = -2$ and d is even. Then there exist scalars $\alpha, h, \sigma, \alpha^*, h^*, \sigma^*, \tau$ in \mathbb{F} such that

$$\begin{aligned}\theta_i &= \begin{cases} \alpha + \sigma + h(i - d/2) & \text{if } i \text{ is even,} \\ \alpha - \sigma - h(i - d/2) & \text{if } i \text{ is odd,} \end{cases} \\ \theta_i^* &= \begin{cases} \alpha^* + \sigma^* + h^*(i - d/2) & \text{if } i \text{ is even,} \\ \alpha^* - \sigma^* - h^*(i - d/2) & \text{if } i \text{ is odd} \end{cases}\end{aligned}$$

for $0 \leq i \leq d$, and

$$\begin{aligned}\varphi_i &= \begin{cases} i(\tau - \sigma h^* - \sigma^* h - h h^*(i - (d+1)/2)) & \text{if } i \text{ is even,} \\ (d-i+1)(\tau + \sigma h^* + \sigma^* h + h h^*(i - (d+1)/2)) & \text{if } i \text{ is odd,} \end{cases} \\ \phi_i &= \begin{cases} i(\tau - \sigma h^* + \sigma^* h + h h^*(i - (d+1)/2)) & \text{if } i \text{ is even,} \\ (d-i+1)(\tau + \sigma h^* - \sigma^* h - h h^*(i - (d+1)/2)) & \text{if } i \text{ is odd} \end{cases}\end{aligned}$$

for $1 \leq i \leq d$.

Note 5.4 (See [17, Remark 15.2].) Referring to Lemma 5.3, $\text{Char}(\mathbb{F})$ is either 0 or greater than $d/2$.

Lemma 5.5 (See [17, Lemma 13.1].) Assume $\beta \neq 2$ and $\beta \neq -2$. Pick a nonzero $q \in \mathbb{F}$ such that $\beta = q + q^{-1}$. Then there exist scalars $\alpha, h, \mu, \alpha^*, h^*, \mu^*, \tau$ in \mathbb{F} such that

$$\begin{aligned}\theta_i &= \alpha + \mu q^i + h q^{d-i}, \\ \theta_i^* &= \alpha^* + \mu^* q^i + h^* q^{d-i}\end{aligned}$$

for $0 \leq i \leq d$, and

$$\begin{aligned}\varphi_i &= (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu \mu^* q^{i-1} - h h^* q^{d-i}), \\ \phi_i &= (q^i - 1)(q^{d-i+1} - 1)(\tau - h \mu^* q^{i-1} - \mu h^* q^{d-i})\end{aligned}$$

for $1 \leq i \leq d$.

Note 5.6 (See [17, Remark 13.2].) Referring to Lemma 5.5, $q^i \neq 1$ for $1 \leq i \leq d$.

6 The parameter array of a Leonard pair that is isomorphic to its opposite

Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Let β be the fundamental parameter and let

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d) \tag{24}$$

be a parameter array of A, A^* . Note that $\text{Char}(\mathbb{F}) \neq 2$ by Lemma 3.4. Also note by Lemma 3.2 that $\theta_i + \theta_{d-i} = 0$ and $\theta_i^* + \theta_{d-i}^* = 0$ for $0 \leq i \leq d$.

Proposition 6.1 *Assume $\beta = 2$. Then there exists a nonzero scalar $s \in \mathbb{F}$ such that*

$$\theta_i = d - 2i \quad (0 \leq i \leq d), \quad (25)$$

$$\theta_i^* = d - 2i \quad (0 \leq i \leq d), \quad (26)$$

$$\varphi_i = i(d - i + 1)(s + s^{-1} - 2) \quad (1 \leq i \leq d), \quad (27)$$

$$\phi_i = i(d - i + 1)(s + s^{-1} + 2) \quad (1 \leq i \leq d), \quad (28)$$

after replacing A, A^* with their nonzero scalar multiples if necessary.

Proof. Let the scalars $\alpha, h, \mu, \alpha^*, h^*, \mu^*, \tau$ be from Lemma 5.1. Observe

$$\theta_0 = \alpha + \mu(-d/2), \quad \theta_d = \alpha + \mu(d - d/2).$$

By this and $\theta_0 + \theta_d = 0$ we find $2\alpha = 0$. This forces $\alpha = 0$ since $\text{Char}(\mathbb{F}) \neq 2$. Observe

$$\theta_1 = \mu(1 - d/2) + h(d - 1), \quad \theta_{d-1} = \mu(d - 1 - d/2) + h(d - 1).$$

By this and $\theta_1 + \theta_{d-1} = 0$ we find $2h(d - 1) = 0$. This forces $h = 0$ by Note 5.2. By these comments, $\theta_i = \mu(i - d/2)$ for $0 \leq i \leq d$. By replacing A with its nonzero scalar multiple if necessary, we may assume $\mu = -2$. So (25) holds. Similarly, $\alpha^* = 0$ and $h^* = 0$, and we may assume $\mu^* = -2$. So (26) holds. Pick a nonzero $s \in \mathbb{F}$ such that $\tau = s + s^{-1}$. Then (27) and (28) hold. \square

Lemma 6.2 *For a nonzero $s \in \mathbb{F}$, define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d$ by (25)–(28). Then (24) is a parameter array over \mathbb{F} if and only if the following (i) and (ii) hold:*

- (i) $\text{Char}(\mathbb{F})$ is either 0 or greater than d .
- (ii) $s^2 \neq 1$.

Proof. First assume (24) is a parameter array over \mathbb{F} .

(i): See Note 5.2.

(ii): If $s^2 = 1$, then $\varphi_1 = 0$ or $\phi_1 = 0$, contradicting Lemma 2.10(ii).

Next assume (i) and (ii) hold. One routinely checks conditions (i)–(v) in Lemma 2.10. So (24) is a parameter array over \mathbb{F} . \square

Proposition 6.3 Assume $\beta = -2$. Then d is even. Moreover, there exists a scalar $\tau \in \mathbb{F}$ such that

$$\theta_i = \begin{cases} 2i - d & \text{if } i \text{ is even,} \\ d - 2i & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq d), \quad (29)$$

$$\theta_i^* = \begin{cases} 2i - d & \text{if } i \text{ is even,} \\ d - 2i & \text{if } i \text{ is odd,} \end{cases} \quad (0 \leq i \leq d), \quad (30)$$

$$\varphi_i = \begin{cases} 2i(d - 2i + 1 + \tau) & \text{if } i \text{ is even,} \\ -2(d - i + 1)(d - 2i + 1 - \tau) & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d), \quad (31)$$

$$\phi_i = \begin{cases} -2i(d - 2i + 1 - \tau) & \text{if } i \text{ is even,} \\ 2(d - i + 1)(d - 2i + 1 + \tau) & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d), \quad (32)$$

after replacing A, A^* with their nonzero scalar multiples if necessary.

Proof. We first show that d is even. By way of contradiction, assume d is odd. Set $m = (d - 1)/2$. By Definition 2.12,

$$\frac{\theta_{m-1} - \theta_{m+2}}{\theta_m - \theta_{m+1}} = \beta + 1 = -1.$$

We have $\theta_m + \theta_{m+1} = 0$ and $\theta_{m-1} + \theta_{m+2} = 0$. By these comments $\theta_m = \theta_{m+2}$, contradicting Lemma 2.10(i). Thus d must be even. Let the scalars $\alpha, h, \sigma, \alpha^*, h^*, \sigma^*, \tau$ be from Lemma 5.3. Observe

$$\theta_0 = \alpha + \sigma + h(-d/2), \quad \theta_d = \alpha + \sigma + h(d - d/2).$$

By this and $\theta_0 + \theta_d = 0$ we find $2(\alpha + \sigma) = 0$. This forces $\alpha + \sigma = 0$ by $\text{Char}(\mathbb{F}) \neq 2$. Observe

$$\theta_1 = \alpha - \sigma - h(1 - d/2), \quad \theta_{d-1} = \alpha - \sigma - h(d - 1 - d/2).$$

By this and $\theta_1 + \theta_{d-1} = 0$ we find $2(\alpha - \sigma) = 0$, so $\alpha - \sigma = 0$. By these comments $\alpha = 0$ and $\sigma = 0$. So $\theta_i = h(i - d/2)$ if i is even, and $\theta_i = -h(i - d/2)$ if i is odd. By replacing A with its scalar multiple if necessary, we may assume $h = 2$. Similarly $\alpha^* = 0$ and $\sigma^* = 0$, and we may assume $h^* = 2$. So (29) and (30) hold. Replacing τ with 2τ we get (31) and (32). \square

Lemma 6.4 Assume d is even. For $\tau \in \mathbb{F}$, define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d$ by (29)–(32). Then (24) is a parameter array over \mathbb{F} if and only if the following (i) and (ii) hold:

- (i) $\text{Char}(\mathbb{F})$ is 0 or greater than d .
- (ii) τ is not among $1 - d, 3 - d, \dots, d - 1$.

Proof. First assume (24) is a parameter array over \mathbb{F} .

(i): Follows from the fact that $\{\theta_i\}_{i=0}^d$ are mutually distinct.

(ii): By way of contradiction, assume that τ is among $1-d, 3-d, \dots, d-1$. So τ is an odd integer such that $1-d \leq \tau \leq d-1$. Set $i = (d+1-\tau)/2$. Observe $d-2i+1-\tau = 0$, and i is an integer such that $1 \leq i \leq d$. Now $\phi_i = 0$ by (32) if i is even, and $\varphi_i = 0$ by (31) if i is odd; contradicting Lemma 2.10(ii).

Next assume (i) and (ii) hold. One routinely checks conditions (i)–(v) in Lemma 2.10. So (24) is a parameter array over \mathbb{F} . \square

Proposition 6.5 *Assume $\beta \neq 2$ and $\beta \neq -2$. Then there exist nonzero scalars $q, s \in \mathbb{F}$ such that*

$$\theta_i = q^i - q^{d-i} \quad (0 \leq i \leq d), \quad (33)$$

$$\theta_i^* = q^i - q^{d-i} \quad (0 \leq i \leq d), \quad (34)$$

$$\varphi_i = (q^i - 1)(q^{d-i+1} - 1)(s - q^{i-1})(s - q^{d-i})s^{-1} \quad (1 \leq i \leq d), \quad (35)$$

$$\phi_i = (q^i - 1)(q^{d-i+1} - 1)(s + q^{i-1})(s + q^{d-i})s^{-1} \quad (1 \leq i \leq d). \quad (36)$$

The scalar q satisfies $\beta = q + q^{-1}$.

Proof. Let the scalars $\alpha, h, \mu, \alpha^*, h^*, \mu^*, \tau$ be from Lemma 5.5. Observe

$$\theta_0 = \alpha + \mu + hq^d, \quad \theta_d = \alpha + \mu q^d + h.$$

By this and $\theta_0 + \theta_d = 0$,

$$2\alpha + (\mu + h)(q^d + 1) = 0. \quad (37)$$

Observe

$$\theta_1 = \alpha + \mu q + hq^{d-1}, \quad \theta_{d-1} = \alpha + \mu q^{d-1} + hq.$$

By this and $\theta_1 + \theta_{d-1} = 0$,

$$2\alpha + (\mu + h)(q + q^{d-1}) = 0. \quad (38)$$

In (37) and (38), eliminate α to find

$$(q - 1)(q^{d-1} - 1)(\mu + h) = 0.$$

By this and Note 5.6 $\mu + h = 0$. By this and (38) $\alpha = 0$. By replacing A with its nonzero scalar multiple if necessary, we may assume $\mu = 1$, and so $h = -1$. Similarly, $\alpha^* = 0$ and $\mu^* + h^* = 0$, and we may assume $\mu^* = 1$ and $h^* = -1$. Pick a nonzero $s \in \mathbb{F}$ such that $\tau = s + s^{-1}q^{d-1}$. Then (33)–(36) hold. By (33) and Definition 2.12 one finds $\beta = q + q^{-1}$. \square

Note 6.6 In Proposition 6.5, the scalar s can be replaced by $s^{-1}q^{d-1}$. Actually, if we replace s with $s^{-1}q^{d-1}$, the values of (35) and (36) are invariant.

Lemma 6.7 For nonzero $q, s \in \mathbb{F}$, define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d$ by (33)–(36). Then (24) is a parameter array over \mathbb{F} if and only if the following (i)–(iii) hold:

- (i) $q^i \neq 1$ for $1 \leq i \leq d$;
- (ii) $q^i \neq -1$ for $0 \leq i \leq d-1$.
- (iii) $s^2 \neq q^{2i}$ for $0 \leq i \leq d-1$.

Proof. First assume (24) is a parameter array over \mathbb{F} .

(i): See Note 5.6.

(ii): Assume $q^i = -1$ for some i ($0 \leq i \leq d-1$). Then $\theta_0 - \theta_{d-i} = (q^i + 1)(1 - q^{d-i}) = 0$ by (33), contradicting Lemma 2.10(i).

(iii): Assume $s^2 = q^{2i}$ for some i ($0 \leq i \leq d-1$). So $s = q^i$ or $s = -q^i$. First assume $s = q^i$. Then $\varphi_{i+1} = 0$ by (35), contradicting Lemma 2.10(ii). Next assume $s = -q^i$. Then $\phi_{i+1} = 0$ by (36), contradicting Lemma 2.10(ii). The result follows.

Next assume (i)–(iii) hold. One routinely checks conditions (i)–(v) in Lemma 2.10. So (24) is a parameter array over \mathbb{F} . \square

Definition 6.8 We define the type of a Leonard pair as follows.

- (i) A, A^* is said to have *Krawtchouk type* whenever $\beta = 2$, $h = 0$ and $h^* = 0$, where h, h^* are from Lemma 5.1.
- (ii) A, A^* is said to have *Bannai/Ito type* whenever $\beta = -2$.
- (iii) A, A^* is said to have *q -Racah type* whenever $\mu \neq 0$, $h \neq 0$, $\mu^* \neq 0$ and $h \neq 0$, where μ, h, μ^*, h^* are from Lemma 5.5.

Proof of Proposition 1.11. Let A, A^* be a Leonard pair on V that is isomorphic to $-A, -A^*$. Let β be the fundamental parameter of A, A^* . First assume $\beta = 2$. Then A, A^* has Krawtchouk type by Proposition 6.1. Next assume $\beta = -2$. Then A, A^* has Bannai/Ito type with even diameter by Proposition 6.3. Next assume $\beta \neq 2$ and $\beta \neq -2$. Then A, A^* has q -Racah type by Proposition 6.5. \square

7 List of zero-diagonal TD-TD Leonard pairs in $\text{Mat}_{d+1}(\mathbb{F})$

In this section, we display five families of zero-diagonal TD-TD Leonard pairs in $\text{Mat}_{d+1}(\mathbb{F})$. In view of Note 1.15, for nonzero scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$, we consider the following zero-diagonal TD-TD pair in $\text{Mat}_{d+1}(\mathbb{F})$:

$$\begin{pmatrix} 0 & z_1 & & & & \mathbf{0} \\ 1 & 0 & z_2 & & & \\ & 1 & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & z_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bar{y}_1 & & & & \mathbf{0} \\ x_1 & 0 & \bar{y}_2 & & & \\ & x_2 & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \bar{y}_d \\ \mathbf{0} & & & & x_d & 0 \end{pmatrix}, \quad (39)$$

where $\bar{y}_i = y_i z_i$ for $1 \leq i \leq d$.

Proposition 7.1 Fix a nonzero $s \in \mathbb{F}$. Assume the conditions (i), (ii) in Lemma 6.2 hold. Consider the pair (39) with

$$\begin{aligned} x_i &= s & (1 \leq i \leq d), \\ y_i &= s^{-1} & (1 \leq i \leq d), \\ z_i &= i(d-i+1) & (1 \leq i \leq d). \end{aligned}$$

Then (39) is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, this Leonard pair has fundamental parameter $\beta = 2$ and parameter array in Proposition 6.1.

Proposition 7.2 Fix $\tau \in \mathbb{F}$ and $\epsilon \in \{1, -1\}$. Assume the conditions (i), (ii) in Lemma 6.4 hold. Consider the pair (39) with

$$\begin{aligned} x_i &= (-1)^{i-1} \epsilon & (1 \leq i \leq d), \\ y_i &= (-1)^{i-1} \epsilon & (1 \leq i \leq d), \\ z_i &= \begin{cases} i(d-i+1-\epsilon\tau) & \text{if } i \text{ is even,} \\ (d-i+1)(i+\epsilon\tau) & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

Then (39) is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, this Leonard pair has fundamental parameter $\beta = -2$ and parameter array in Proposition 6.3.

Proposition 7.3 Fix nonzero scalars $q, s \in \mathbb{F}$. Assume the conditions (i)–(iii) in Lemma 6.7 hold. Consider the pair (39) with

$$\begin{aligned} x_i &= sq^{1-i} & (1 \leq i \leq d), \\ y_i &= s^{-1}q^{d-i} & (1 \leq i \leq d), \\ z_i &= q^{i-1}(q^i-1)(q^{d-i+1}-1). & (1 \leq i \leq d). \end{aligned}$$

Then (39) is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, this Leonard pair has fundamental parameter $\beta = q + q^{-1}$ and parameter array in Proposition 6.5.

Proposition 7.4 Fix nonzero scalars $q, s \in \mathbb{F}$. Assume the conditions (i)–(iii) in Lemma 6.7 hold. Also assume $s^2 \neq q^i$ for $0 \leq i \leq 2d-2$. Consider the pair (39) with

$$\begin{aligned} x_i &= sq^{1-i} & (1 \leq i \leq d), \\ y_i &= s^{-1}q^{i-1} & (1 \leq i \leq d), \\ z_1 &= \frac{(q-1)(q^d-1)(s^2-q^d)}{s^2-q}, \\ z_i &= \frac{q^{i-1}(q^i-1)(q^{d-i+1}-1)(s^2-q^{i-2})(s^2-q^{d+i-1})}{(s^2-q^{2i-3})(s^2-q^{2i-1})} & (2 \leq i \leq d-1), \\ z_d &= \frac{q^{d-1}(q-1)(q^d-1)(s^2-q^{d-2})}{s^2-q^{2d-3}}. \end{aligned}$$

Then (39) is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, this Leonard pair has fundamental parameter $\beta = q + q^{-1}$ and parameter array in Proposition 6.5.

Proposition 7.5 Fix nonzero scalars $q, s \in \mathbb{F}$. Assume d is even, and the conditions (i)–(iii) in Lemma 6.7 hold. Consider the pair (39) with

$$\begin{aligned} x_i &= sq^{1-i} & (1 \leq i \leq d), \\ y_i &= sq^{1-i} & (1 \leq i \leq d), \\ z_i &= \begin{cases} q^d(q^i - 1)(1 - s^{-2}q^{i-2}) & \text{if } i \text{ is even,} \\ -q^{i-1}(q^{d-i+1} - 1)(1 - s^{-2}q^{d+i-1}) & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

Then (39) is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Moreover, this Leonard pair has fundamental parameter $\beta = q + q^{-1}$ and parameter array in Proposition 6.5.

8 Askey-Wilson relations

In this section we recall the Askey-Wilson relations for a Leonard pair. Let A, A^* be a Leonard pair on V with parameter array

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

and fundamental parameter β .

Lemma 8.1 (See [10, Theorem 11.1].) *There exist scalars $\gamma, \gamma^*, \varrho, \varrho^*$ such that*

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \quad (1 \leq i \leq d-1), \quad (40)$$

$$\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq d-1), \quad (41)$$

$$\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d), \quad (42)$$

$$\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \quad (1 \leq i \leq d). \quad (43)$$

Let the scalars $\gamma, \gamma^*, \varrho, \varrho^*$ be as in Lemma 8.1.

Lemma 8.2 (See [25, Theorem 1.5].) *There exist scalars ω, η, η^* such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \quad (44)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \quad (45)$$

The scalars ω, η, η^* are uniquely determined by A, A^* .

The relations (44) and (45) are known as the *Askey-Wilson relations*. Below we describe the scalars ω, η, η^* . For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*). Let the scalars $\{a_i\}_{i=0}^d, \{a_i^*\}_{i=0}^d$ be from Definition 2.14. For notational convenience, define $\theta_{-1}, \theta_{d+1}$ (resp. $\theta_{-1}^*, \theta_{d+1}^*$) so that (40) (resp. (41)) holds for $i = 0$ and $i = d$. Let the scalars ω, η, η^* be from Lemma 8.2.

Lemma 8.3 (See [25, Theorem 5.3].) *With the above notation,*

$$\begin{aligned}\omega &= a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_{i-1} + \theta_i) & (1 \leq i \leq d), \\ \eta &= a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i & (0 \leq i \leq d), \\ \eta^* &= a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma\theta_i^{*2} - \omega\theta_i^* & (0 \leq i \leq d).\end{aligned}$$

We mention a lemma for later use.

Lemma 8.4 (See [20, Proof of Theorem 3.10].) *Consider linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (44) for some scalars $\beta, \gamma, \gamma^*, \varrho, \omega, \eta, \eta^* \in \mathbb{F}$. Assume A is multiplicity-free with eigenvalues $\{\theta_i\}_{i=0}^d$. For $0 \leq i \leq d$ let E_i be the primitive idempotent of A associated with θ_i . Assume that for $0 \leq i, j \leq d$*

$$\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho \neq 0 \quad \text{if } |i - j| > 1.$$

Then $E_i A^ E_j = 0$ if $|i - j| > 1$ for $0 \leq i, j \leq d$.*

Below we obtain the scalars $\gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ for a Leonard pair that is isomorphic to its opposite.

Lemma 8.5 *Assume $\beta = 2$, and the parameter array satisfies (25)–(28) for a nonzero $s \in \mathbb{F}$. Then $\gamma = 0, \gamma^* = 0, \eta = 0, \eta^* = 0$, and*

$$\varrho = 4, \quad \varrho^* = 4, \quad \omega = -2(s + s^{-1}).$$

Lemma 8.6 *Assume $\beta = -2$, and the parameter array satisfies (29)–(32) for a scalar $\tau \in \mathbb{F}$. Then $\gamma = 0, \gamma^* = 0, \eta = 0, \eta^* = 0$, and*

$$\varrho = 4, \quad \varrho^* = 4, \quad \omega = 4(d + 1)\tau.$$

Lemma 8.7 *Assume $\beta \neq 2, \beta \neq -2$, and pick a nonzero $q \in \mathbb{F}$ such that $\beta = q + q^{-1}$. Assume the parameter array satisfies (33)–(36) for a nonzero $s \in \mathbb{F}$. Then $\gamma = 0, \gamma^* = 0, \eta = 0, \eta^* = 0$, and*

$$\begin{aligned}\varrho &= q^{d-2}(q^2 - 1)^2, & \varrho^* &= q^{d-2}(q^2 - 1)^2, \\ \omega &= -q^{-1}(q - 1)^2(q^{d+1} + 1)(s + s^{-1}q^{d-1}).\end{aligned}$$

9 The characteristic polynomial of a zero-diagonal TD matrix

In this section we display a formula for the characteristic polynomial of a zero-diagonal TD matrix. Let $A \in \text{Mat}_{d+1}(\mathbb{F})$ be a zero-diagonal TD matrix. In view of Note 1.15, there

exists an invertible diagonal matrix $D \in \text{Mat}_{d+1}(\mathbb{F})$ such that $D^{-1}AD$ has all subdiagonal entries 1. Clearly A and $D^{-1}AD$ has the same characteristic polynomial. So we assume

$$A = \begin{pmatrix} 0 & z_1 & & & \mathbf{0} \\ 1 & 0 & z_2 & & \\ & 1 & 0 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & z_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}.$$

Definition 9.1 For an integer i with $0 \leq i \leq \lfloor (d+1)/2 \rfloor$, let $\pi(d, i)$ denote the sum of $z_{\ell_1} z_{\ell_2} \cdots z_{\ell_i}$ over all $(\ell_1, \ell_2, \dots, \ell_i)$ such that $1 \leq \ell_1, \ell_2, \dots, \ell_i \leq d$ and $\ell_{j+1} - \ell_j \geq 2$ for $1 \leq j \leq i-1$.

Let $f(x)$ be the characteristic polynomial of A :

$$f(x) = \det(xI - A).$$

Then

$$f(x) = \sum_{i=0}^{\lfloor (d+1)/2 \rfloor} (-1)^i \pi(d, i) x^{d-2i+1}. \quad (46)$$

The proof of (46) is routine using induction on d .

Example 9.2 When $d = 5$,

$$\begin{aligned} f(x) &= x^6 - x^4(z_1 + z_2 + z_3 + z_4 + z_5) \\ &\quad + x^2(z_1 z_3 + z_1 z_4 + z_1 z_5 + z_2 z_4 + z_2 z_5 + z_3 z_5) - z_1 z_3 z_5. \end{aligned}$$

When $d = 6$,

$$\begin{aligned} f(x) &= x^7 - x^5(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ &\quad + x^3(z_1 z_3 + z_1 z_4 + z_1 z_5 + z_1 z_6 + z_2 z_4 + z_2 z_5 + z_2 z_6 + z_3 z_5 + z_3 z_6 + z_4 z_6) \\ &\quad - x(z_1 z_3 z_5 + z_1 z_3 z_6 + z_1 z_4 z_6 + z_2 z_4 z_6). \end{aligned}$$

10 Proof of Proposition 7.1

Fix a nonzero $s \in \mathbb{F}$, and assume conditions (i), (ii) in Lemma 6.2 hold. Define scalars $\{x_i\}_{i=1}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ as in Proposition 7.1, and let A, A^* be the zero-diagonal TD-TD pair (39). Define scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ by (25), (26).

Lemma 10.1 *The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct. Moreover A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$).*

Proof. By Lemma 6.2(i) the scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. Using (46) one checks that $\det(\theta_i I - A) = 0$ for $0 \leq i \leq d$. So θ_i is a root of the characteristic polynomial of A . Therefore $\{\theta_i\}_{i=0}^d$ are the eigenvalues of A . The proof for A^* is similar. \square

Define $A^\varepsilon \in \text{Mat}_{d+1}(\mathbb{F})$ by

$$A^\varepsilon = AA^* - A^*A. \quad (47)$$

Define scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ by

$$\theta_i^\varepsilon = (d - 2i)(s - s^{-1}) \quad (0 \leq i \leq d).$$

Lemma 10.2 *The scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct. Moreover*

$$A^\varepsilon = \text{diag}(\theta_0^\varepsilon, \theta_1^\varepsilon, \dots, \theta_d^\varepsilon). \quad (48)$$

Proof. The scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct since $s - s^{-1} \neq 0$ by Lemma 6.2(ii). One routinely checks (48). \square

Lemma 10.3 *The matrices A, A^*, A^ε satisfy*

$$A^*A^\varepsilon - A^\varepsilon A^* = -4A + 2(s + s^{-1})A^*, \quad (49)$$

$$A^\varepsilon A - AA^\varepsilon = -4A^* + 2(s + s^{-1})A. \quad (50)$$

Proof. Routine verification. \square

Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ be as in Lemma 8.5.

Lemma 10.4 *The matrices A, A^* satisfy (44) and (45).*

Proof. In (49) and (50), eliminate A^ε using (47). \square

For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 10.5 *For $0 \leq i, j \leq d$ such that $|i - j| > 1$,*

$$E_i A^* E_j = 0, \quad E_i^* A E_j^* = 0.$$

Proof. We have $\gamma = 0$ and $\varrho = 4$. By this and (25),

$$\theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho = 4(i - j - 1)(i - j + 1) \neq 0.$$

Now $E_i A^* E_j = 0$ by Lemma 8.4. The proof of $E_i^* A E_j^* = 0$ is similar. \square

For $0 \leq i \leq d$ let E_i^ε be the primitive idempotent of A^ε associate with θ_i^ε . Consider the sequence

$$\Phi^\varepsilon = (A, \{E_i\}_{i=0}^d, A^\varepsilon, \{E_i^\varepsilon\}_{i=0}^d).$$

Lemma 10.6 Φ^ε is a Leonard system.

Proof. We verify conditions (i)–(v) in Definition 2.1. By Lemmas 10.1 and 10.2 each of A , A^ε is multiplicity-free, so condition (i) holds. By the construction, conditions (ii), (iii) hold. Concerning condition (iv), pick integers i, j such that $0 \leq i, j \leq d$ and $|i - j| > 1$. By the shape of A we have $E_i^\varepsilon A E_j^\varepsilon = 0$. We show $E_i A^\varepsilon E_j = 0$. In (50), multiply each side on the left by E_i and on the right by E_j to find

$$(\theta_j - \theta_i) E_i A^\varepsilon E_j = -4 E_i A^* E_j.$$

By Lemma 10.5 $E_i A^* E_j = 0$. By these comments $E_i A^\varepsilon E_j = 0$. Thus condition (iv) holds. Concerning condition (v), pick integers i, j such that $0 \leq i, j \leq d$ and $|i - j| = 1$. We have $E_i^\varepsilon A E_j^\varepsilon \neq 0$ by the shape of A . Now apply Lemma 2.18 to Φ^ε to find that Φ^ε is a Leonard system. \square

Lemma 10.7 A and A^* together generate $\text{Mat}_{d+1}(\mathbb{F})$.

Proof. By Lemmas 2.18 and 10.6 A^ε and A together generate $\text{Mat}_{d+1}(\mathbb{F})$. By (47) A^ε is a polynomial in A and A^* . By these comments A and A^* together generate $\text{Mat}_{d+1}(\mathbb{F})$. \square

Proof of Proposition 7.1. Consider the sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$. We check conditions (i)–(v) in Definition 2.1. By Lemma 10.1 each of A, A^* is multiplicity-free, so condition (i) holds. By the construction conditions (ii) and (iii) holds. By Lemma 10.5 condition (iv) holds. By Lemmas 2.18 and 10.7 condition (v) holds. Thus Φ is a Leonard system, and so A, A^* is a Leonard pair. Concerning the parameter array of A, A^* , define $\{\varphi_i\}_{i=1}^d$ and $\{\phi_i\}_{i=1}^d$ by (3) and (4). One routinely checks that

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

coincides with the parameter array in Proposition 6.1. \square

11 Proof of Proposition 7.2

Fix $\tau \in \mathbb{F}$, and assume conditions (i), (ii) in Lemma 6.4 hold. Note that d is even and $\text{Char}(\mathbb{F}) \neq 2$. Fix $\epsilon \in \{1, -1\}$, and define scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.2. Let A, A^* be the zero-diagonal TD-TD pair (39). Define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ by (29), (30).

Lemma 11.1 The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct. Moreover A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$).

Proof. By Lemma 6.4(i) the scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. Using (46) one checks that $\det(\theta_i I - A) = 0$ for $0 \leq i \leq d$. So θ_i is a root of the characteristic polynomial of A . Therefore $\{\theta_i\}_{i=0}^d$ are the eigenvalues of A . The proof for A^* is similar. \square

Define $A^\varepsilon \in \text{Mat}_{d+1}(\mathbb{F})$ by

$$A^\varepsilon = AA^* + A^*A. \quad (51)$$

Define scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ by

$$\theta_i^\varepsilon = \begin{cases} 2d\tau + 2(d-2i)\epsilon & \text{if } i \text{ is even,} \\ (2d+4)\tau - 2(d-2i)\epsilon & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq d).$$

Lemma 11.2 *The scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct. Moreover*

$$A^\varepsilon = \text{diag}(\theta_0^\varepsilon, \theta_1^\varepsilon, \dots, \theta_d^\varepsilon). \quad (52)$$

Proof. The scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct by conditions (i), (ii) in Lemma 6.4. One routinely checks (48). \square

Lemma 11.3 *The matrices A, A^*, A^ε satisfy*

$$A^*A^\varepsilon + A^\varepsilon A^* = 4A + 4(d+1)\tau A^*, \quad (53)$$

$$A^\varepsilon A + AA^\varepsilon = 4A^* + 4(d+1)\tau A. \quad (54)$$

Proof. Routine verification. \square

Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ be as in Lemma 8.6.

Lemma 11.4 *The matrices A, A^* satisfy (44) and (45).*

Proof. In (53) and (54), eliminate A^ε using (51). \square

For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 11.5 *For $0 \leq i, j \leq d$ such that $|i - j| > 1$,*

$$E_i A^* E_j = 0, \quad E_i^* A E_j^* = 0.$$

Proof. We have $\gamma = 0$ and $\varrho = 4$, so

$$\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho = (\theta_i + \theta_j - 2)(\theta_i + \theta_j + 2).$$

Using (29)

$$\theta_i + \theta_j = \begin{cases} 2(i+j-d) & \text{if } i \text{ is even, } j \text{ is even,} \\ 2(i-j) & \text{if } i \text{ is even, } j \text{ is odd,} \\ 2(j-i) & \text{if } i \text{ is odd, } j \text{ is even,} \\ 2(d-i-j) & \text{if } i \text{ is odd, } j \text{ is odd} \end{cases} \quad (0 \leq i, j \leq d).$$

Using this and condition (i) in Lemma 6.4, one checks $\theta_i + \theta_j - 2 \neq 0$ and $\theta_i + \theta_j + 2 \neq 0$ if $|i - j| > 1$. By this and Lemma 8.4 $E_i A^* E_j = 0$. The proof of $E_i^* A E_j^* = 0$ is similar. \square

For $0 \leq i \leq d$ let E_i^ε be the primitive idempotent of A^ε associate with θ_i^ε . Consider the sequence

$$\Phi^\varepsilon = (A, \{E_i\}_{i=0}^d, A^\varepsilon, \{E_i^\varepsilon\}_{i=0}^d).$$

Lemma 11.6 Φ^ε is a Leonard system.

Proof. Similar to the proof of Lemma 10.6. \square

Lemma 11.7 The matrices A and A^* together generate $\text{Mat}_{d+1}(\mathbb{F})$.

Proof. Similar to the proof of Lemma 10.7. \square

Proof of Proposition 7.2. Consider the sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$. We check conditions (i)–(v) in Definition 2.1. By Lemma 11.1 each of A, A^* is multiplicity-free, so condition (i) holds. By the construction conditions (ii) and (iii) holds. By Lemma 11.5 condition (iv) holds. By Lemmas 2.18 and 11.7 conditions (v) holds. Thus Φ is a Leonard system, and so A, A^* is a Leonard pair. One can show that A, A^* has parameter array in Proposition 6.3 in a similar way as in the proof of Proposition 7.1. \square

12 Proof of Proposition 7.4

Fix a nonzero $q, s \in \mathbb{F}$, and assume conditions (i)–(iii) in Lemma 6.7 hold. Also assume

$$s^2 \neq q^i \quad (0 \leq i \leq 2d - 2). \quad (55)$$

Define scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.4, and let A, A^* be the zero-diagonal TD-TD pair (39). Define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ by (33), (34).

Lemma 12.1 The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct. Moreover A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$).

Proof. By conditions (i), (ii) in Lemma 6.7 the scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. Using (46) one checks that $\det(\theta_i I - A) = 0$ for $0 \leq i \leq d$. So θ_i is a root of the characteristic polynomial of A . Therefore $\{\theta_i\}_{i=0}^d$ are the eigenvalues of A . The proof for A^* is similar. \square

Define $A^\varepsilon \in \text{Mat}_{d+1}(\mathbb{F})$ by

$$A^\varepsilon = A A^* - q A^* A. \quad (56)$$

Define scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ by

$$\theta_i^\varepsilon = q^{d-i}(q^2 - 1)(s + s^{-1}q^{2i-1}) - (q - 1)(q^{d+1} + 1)(s + s^{-1}q^{d-1}) \quad (0 \leq i \leq d).$$

Lemma 12.2 *The scalars $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct. Moreover*

$$A^\varepsilon = \text{diag}(\theta_0^\varepsilon, \theta_1^\varepsilon, \dots, \theta_d^\varepsilon). \quad (57)$$

Proof. For $0 \leq i < j \leq d$

$$\theta_i^\varepsilon - \theta_j^\varepsilon = s^{-1}q^{d-j}(q^2 - 1)(q^{j-i} - 1)(s^2 - q^{i+j-1}).$$

In this line, the right-hand side is nonzero by Lemma 6.7(i) and (55). So $\{\theta_i^\varepsilon\}_{i=0}^d$ are mutually distinct. One routinely checks (57). \square

Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ be as in Lemma 8.7.

Lemma 12.3 *The matrices A, A^*, A^ε satisfy*

$$A^*A^\varepsilon - qA^\varepsilon A^* = -q\varrho^*A - q\omega A^*, \quad (58)$$

$$A^\varepsilon A - qAA^\varepsilon = -q\varrho A^* - q\omega A. \quad (59)$$

Proof. Routine verification. \square

Lemma 12.4 *The matrices A, A^* satisfy (44) and (45).*

Proof. In (58) and (59), eliminate A^ε using (56). \square

For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 12.5 *For $0 \leq i, j \leq d$ such that $|i - j| > 1$,*

$$E_i A^* E_j = 0, \quad E_i^* A E_j^* = 0.$$

Proof. We have $\gamma = 0$ and $\varrho = q^{d-2}(q^2 - 1)^2$. By this and (33),

$$\begin{aligned} \theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho \\ = q^{2j}(q^{i-j+1} - 1)(q^{i-j-1} - 1)(q^{d-i-j-1} + 1)(q^{d-i-j+1} + 1). \end{aligned}$$

In this equation, the right-hand side is nonzero by conditions (i), (ii) in Lemma 6.7. So $E_i A^* E_j = 0$ by Lemma 8.4. The proof of $E_i^* A E_j^* = 0$ is similar. \square

For $0 \leq i \leq d$ let E_i^ε be the primitive idempotent of A^ε associate with θ_i^ε . Consider the sequence

$$\Phi^\varepsilon = (A, \{E_i\}_{i=0}^d, A^\varepsilon, \{E_i^\varepsilon\}_{i=0}^d).$$

Lemma 12.6 *Φ^ε is a Leonard system.*

Proof. Similar to the proof of Lemma 10.6. \square

Lemma 12.7 *The matrices A and A^* together generate $\text{Mat}_{d+1}(\mathbb{F})$.*

Proof. Similar to the proof of Lemma 10.7. \square

Proof of Proposition 7.4. Consider the sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$. We check conditions (i)–(v) in Definition 2.1. By Lemma 12.1 each of A, A^* is multiplicity-free, so condition (i) holds. By the construction conditions (ii) and (iii) hold. By Lemma 12.5 condition (iv) holds. By Lemmas 2.18 and 12.7 condition (v) holds. Thus Φ is a Leonard system, and so A, A^* is a Leonard pair. One can show that A, A^* has parameter array in Proposition 6.5 in a similar way as in the proof of Proposition 7.1. \square

13 Proof of Proposition 7.3

Fix a nonzero $q, s \in \mathbb{F}$, and assume conditions (i)–(iii) in Lemma 6.7 hold. Define scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.3, and let A, A^* be the zero-diagonal TD-TD pair (39). Define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ by (33), (34).

Lemma 13.1 *The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct. Moreover A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$).*

Proof. Similar to the proof of Lemma 12.1. \square

Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ be as in Lemma 8.7.

Lemma 13.2 *The matrices A, A^* satisfy (44) and (45).*

Proof. Routine verification. \square

For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 13.3 *For $0 \leq i, j \leq d$ such that $|i - j| > 1$,*

$$E_i A^* E_j = 0, \quad E_i^* A E_j^* = 0.$$

Proof. Similar to the proof of Lemma 12.5. \square

For $a \in \mathbb{F}$ and an integer $n \geq 0$, define

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}).$$

We interpret $(a; q)_0 = 1$.

Lemma 13.4 For $1 \leq r \leq d$

$$E_{r-1}A^*E_r \neq 0, \quad E_rA^*E_{r-1} \neq 0.$$

Proof. One routinely checks that for $1 \leq r \leq d$,

$$(E_{r-1}A^*E_r)_{0,0} = \frac{sq^{d-2r+1}(1-s^{-2}q^{2r-2})(q;q)_d}{2(1+q^{d-2r+1})(q^2;q^2)_{r-1}(q^2;q^2)_{d-r}},$$

$$(E_rA^*E_{r-1})_{0,0} = -\frac{s(1-s^{-2}q^{2d-2r})(q;q)_d}{2(1+q^{d-2r+1})(q^2;q^2)_{r-1}(q^2;q^2)_{d-r}}.$$

By this and using conditions (i)–(iii) in Lemma 6.7 one finds $(E_{r-1}A^*E_r)_{0,0} \neq 0$ and $(E_rA^*E_{r-1})_{0,0} \neq 0$. The result follows. \square

Proof of Proposition 7.3. Consider the sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$. We check conditions (i)–(v) in Definition 2.1. By Lemma 13.1 each of A, A^* is multiplicity-free, so condition (i) holds. By the construction conditions (ii) and (iii) hold. By Lemmas 13.3 conditions (iv) holds. By Lemmas 2.18 and 13.4 condition (v) holds. Thus Φ is a Leonard system, and so A, A^* is a Leonard pair. One can show that A, A^* has parameter array in Proposition 6.5 in a similar way as in the proof of Proposition 7.1. \square

14 Proof of Proposition 7.5

Assume d is even. Fix a nonzero $q, s \in \mathbb{F}$, and assume conditions (i)–(iii) in Lemma 6.7 hold. Define scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.5, and let A, A^* be the zero-diagonal TD-TD pair (39). Define scalars $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ by (33), (34).

Lemma 14.1 The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct. Moreover A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$).

Proof. Similar to the proof of Lemma 12.1. \square

Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ be as in Lemma 8.7.

Lemma 14.2 The matrices A, A^* satisfy (44) and (45).

Proof. Routine verification. \square

For $0 \leq i \leq d$ let E_i (resp. E_i^*) be the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 14.3 For $0 \leq i, j \leq d$ such that $|i - j| > 1$,

$$E_iA^*E_j = 0, \quad E_i^*AE_j^* = 0.$$

Proof. Similar to the proof of Lemma 12.5. \square

Lemma 14.4 For $1 \leq r \leq d$

$$E_{r-1}A^*E_r \neq 0, \quad E_rA^*E_{r-1} \neq 0.$$

Proof. One routinely checks that for $1 \leq r \leq d$,

$$(E_{r-1}A^*E_r)_{0,0} = \frac{(-1)^{r-1}sq^{r(d-r)}(q^2; q^2)_{d/2}(s^{-2}; q^2)_r(s^{-2}; q^2)_{d-r+1}}{2(1+q^{d-2r+1})(q^2; q^2)_{r-1}(q^2; q^2)_{d-r}(s^{-2}; q^2)_{d/2}},$$

$$(E_rA^*E_{r-1})_{0,0} = \frac{(-1)^{r-1}sq^{r(d-r)}(q^2; q^2)_{d/2}(s^{-2}; q^2)_r(s^{-2}; q^2)_{d-r+1}}{2(1+q^{d-2r+1})(q^2; q^2)_{r-1}(q^2; q^2)_{d-r}(s^{-2}; q^2)_{d/2}}.$$

By this and using conditions (i)–(iii) in Lemma 6.7 one finds $(E_{r-1}A^*E_r)_{0,0} \neq 0$ and $(E_rA^*E_{r-1})_{0,0} \neq 0$. The result follows. \square

Proof of Proposition 7.5. Consider the sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$. We check conditions (i)–(v) in Definition 2.1. By Lemma 14.1, each of A, A^* is multiplicity-free, so condition (i) holds. By the construction conditions (ii) and (iii) hold. By Lemma 14.3 condition (iv) holds. By Lemmas 2.18 and 14.4 condition (v) holds. Thus Φ is a Leonard system, and so A, A^* is a Leonard pair. One can show that A, A^* has parameter array in Proposition 6.5 in a similar way as in the proof of Proposition 7.1. \square

15 Proof of Proposition 1.12

Proof of Proposition 1.12. Let A, A^* be a Leonard pair on V that is isomorphic to its opposite. Let β be the fundamental parameter of A, A^* , and let

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

be a parameter array of A, A^* .

(i): By replacing A, A^* with their nonzero scalar multiples if necessary, we may assume that the parameter array is as in Proposition 6.1. Let B, B^* be the zero-diagonal TD-TD pair (39) in $\text{Mat}_{d+1}(\mathbb{F})$ with the values of $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.1. We show that A, A^* is represented by B, B^* . By Proposition 7.1 the parameter array of B, B^* is as in Proposition 6.1. So A, A^* and B, B^* have the same parameter array, and therefore A, A^* and B, B^* are isomorphic by Lemma 2.9. Thus A, A^* is represented by B, B^* .

(ii), (iii): Similar. \square

16 Evaluating the Askey-Wilson relations

For nonzero scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$, consider the zero-diagonal TD-TD pair (39) in $\text{Mat}_{d+1}(\mathbb{F})$; denote this pair by A, A^* . Assume A, A^* be a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with fundamental parameter β . By Note 1.9 A, A^* is isomorphic to its opposite. Let

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

be a parameter array of A, A^* . We consider the Askey-Wilson relations for A, A^* . Let the scalars $\gamma, \gamma^*, \varrho, \varrho^*$ be from Lemma 8.1, and the scalars ω, η, η^* be from Lemma 8.3. By Lemmas 8.5–8.7 we have $\gamma = 0, \gamma^* = 0, \eta = 0, \eta^* = 0$. So the Askey-Wilson relations (44), (45) become

$$A^2 A^* - \beta A A^* A + A^* A^2 - \varrho A^* - \omega A = 0, \quad (60)$$

$$A^{*2} A - \beta A^* A A^* + A A^{*2} - \varrho^* A - \omega A^* = 0. \quad (61)$$

By (39), for $0 \leq i, j \leq d$

$$A_{i,j} = \begin{cases} 1 & \text{if } i - j = 1, \\ z_j & \text{if } j - i = 1, \\ 0 & \text{if } |i - j| \neq 1, \end{cases} \quad A_{i,j}^* = \begin{cases} x_i & \text{if } i - j = 1, \\ y_j z_j & \text{if } j - i = 1, \\ 0 & \text{if } |i - j| \neq 1. \end{cases}$$

For notational convenience, set $x_i = 0, y_i = 0, z_i = 0$ for $i \leq 0$ or $i > d$.

Lemma 16.1 *Assume A, A^* satisfies (60). Then for $2 \leq i \leq d - 1$*

$$x_{i-1} - \beta x_i + x_{i+1} = 0, \quad (62)$$

$$y_{i-1} - \beta y_i + y_{i+1} = 0. \quad (63)$$

Proof. Compute the $(i+1, i-2)$ -entry of (60) to get (62). Compute the $(i-2, i+1)$ -entry of (60) to get (63). \square

Lemma 16.2 *Assume A, A^* satisfies (61). Then for $2 \leq i \leq d - 1$*

$$x_{i-1} x_i - \beta x_{i-1} x_{i+1} + x_i x_{i+1} = 0, \quad (64)$$

$$y_{i-1} y_i - \beta y_{i-1} y_{i+1} + y_i y_{i+1} = 0. \quad (65)$$

Proof. Compute the $(i+1, i-2)$ -entry of (61) to get (64). Compute the $(i-2, i+1)$ -entry of (61) to get (65). \square

Lemma 16.3 *Assume A, A^* satisfies (60). Then for $1 \leq i \leq d$*

$$\begin{aligned} & z_{i-1}(x_i - \beta x_{i-1} + y_{i-1}) + z_i(2x_i - \beta y_i) \\ & + z_{i+1}(x_i - \beta x_{i+1} + y_{i+1}) - \varrho x_i - \omega = 0, \end{aligned} \quad (66)$$

$$\begin{aligned} & z_{i-1}(y_i - \beta y_{i-1} + x_{i-1}) + z_i(2y_i - \beta x_i) \\ & + z_{i+1}(y_i - \beta y_{i+1} + x_{i+1}) - \varrho y_i - \omega = 0. \end{aligned} \quad (67)$$

Proof. Compute the $(i, i-1)$ -entry of (60) to get (66). Compute the $(i-1, i)$ -entry of (60) to get (67). \square

Lemma 16.4 Assume A, A^* satisfies (61). Then for $1 \leq i \leq d$

$$z_{i-1}(y_{i-1}x_{i-1} - \beta y_{i-1}x_i + x_{i-1}x_i) + z_i(2y_ix_i - \beta x_i^2) + z_{i+1}(y_{i+1}x_{i+1} - \beta y_{i+1}x_i + x_ix_{i+1}) - \varrho^* - \omega x_i = 0, \quad (68)$$

$$z_{i-1}(x_{i-1}y_{i-1} - \beta x_{i-1}y_i + y_{i-1}y_i) + z_i(2x_iy_i - \beta y_i^2) + z_{i+1}(x_{i+1}y_{i+1} - \beta x_{i+1}y_i + y_iy_{i+1}) - \varrho^* - \omega y_i = 0. \quad (69)$$

Proof. Compute the $(i, i-1)$ -entry of (61) to get (68). Compute the $(i-1, i)$ -entry of (61) to get (69). \square

17 Some equations

For nonzero scalars $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ in \mathbb{F} , define $A, A^* \in \text{Mat}_{d+1}(\mathbb{F})$ by

$$A = \begin{pmatrix} 0 & z_1 & & & \mathbf{0} \\ 1 & 0 & z_2 & & \\ & 1 & 0 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & z_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & \bar{y}_1 & & & \mathbf{0} \\ x_1 & 0 & \bar{y}_2 & & \\ & x_2 & 0 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \bar{y}_d \\ \mathbf{0} & & & & x_d & 0 \end{pmatrix},$$

where $\bar{y}_i = y_iz_i$ for $0 \leq i \leq d$. Assume A, A^* is a Leonard pair with parameter array

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d). \quad (70)$$

By Lemma 2.4 there exists a basis for \mathbb{F}^{d+1} , with respect to which the matrices representing A, A^* are

$$A : \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \theta_d \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix},$$

Denote the above matrices by B, B^* . By the construction, there exists an invertible matrix $P \in \text{Mat}_{d+1}(\mathbb{F})$ such that both $AP = PB$ and $A^*P = PB^*$. To simplify notation, define $P_{i,j} = 0$ if i or j is not in $\{0, 1, \dots, d\}$.

Lemma 17.1 For $0 \leq i, j \leq d$,

$$P_{i-1,j} + z_{i+1}P_{i+1,j} - \theta_j P_{i,j} - P_{i,j+1} = 0, \quad (71)$$

$$x_i P_{i-1,j} + y_{i+1}z_{i+1}P_{i+1,j} - \theta_j^* P_{i,j} - \varphi_j P_{i,j-1} = 0. \quad (72)$$

Proof. Compute the (i, j) -entry of $AP - PB$ and $A^*P - PB^*$. \square

Lemma 17.2 *We have $P_{0,0} \neq 0$ and $P_{d,d} \neq 0$.*

Proof. By (72) for $j = 0$,

$$y_{i+1}z_{i+1}P_{i+1,0} = \theta_0^*P_{i,0} - x_iP_{i-1,0} \quad (1 \leq i \leq d).$$

Solving this recursion, we find that $P_{i,0}$ is a scalar multiple of $P_{0,0}$ for $0 \leq i \leq d$. So, if $P_{0,0} = 0$, then 0th column of P is zero; this contradicts that P is invertible. Therefore $P_{0,0} \neq 0$. By (71) for $j = d$,

$$P_{i-1,d} = \theta_d P_{i,d} - z_{i+1}P_{i+1,d} \quad (0 \leq i \leq d).$$

Solving this recursion, we find that $P_{i,d}$ is a scalar multiple of $P_{d,d}$ for $0 \leq i \leq d$. So $P_{d,d} \neq 0$. \square

Lemma 17.3 *We have*

$$y_1^2 z_1(x_1 - y_2) + y_1 y_2(\varphi_1 + \theta_0(\theta_0^* - \theta_1^*)) - \theta_0^*(\theta_0^* y_1 - \theta_1^* y_2) = 0, \quad (73)$$

$$z_d(x_{d-1} - y_d) + \varphi_d + \theta_d^*(\theta_d - \theta_{d-1}) + \theta_d(\theta_{d-1}x_d - \theta_d x_{d-1}) = 0. \quad (74)$$

Proof. We first show (73). By (72) for $(i, j) = (0, 0)$, $(1, 0)$, $(0, 1)$ and (71) for $(i, j) = (0, 0)$, $(1, 0)$,

$$\begin{aligned} -\theta_0^*P_{0,0} + y_1 z_1 P_{1,0} &= 0, \\ x_1 P_{0,0} - \theta_0^* P_{1,0} + y_2 z_2 P_{2,0} &= 0, \\ -\varphi_1 P_{0,0} - \theta_1^* P_{0,1} + y_1 z_1 P_{1,1} &= 0, \\ -\theta_0 P_{0,0} - P_{0,1} + z_1 P_{1,0} &= 0, \\ P_{0,0} - \theta_0 P_{1,0} - P_{1,1} + z_2 P_{2,0} &= 0. \end{aligned}$$

In these equation, eliminate $P_{1,0}$, $P_{0,1}$, $P_{1,1}$, $P_{2,0}$ to find that $P_{0,0}$ times the left-hand side of (73) is zero. By this and Lemma 17.2 we get (73). Next we show (74). By (71) for $(i, j) = (d, d)$, $(d-1, d)$, $(d, d-1)$ and (72) for $(i, j) = (d, d)$, $(d-1, d)$,

$$\begin{aligned} P_{d-1,d} - \theta_d P_{d,d} &= 0, \\ P_{d-2,d} - \theta_d P_{d-1,d} + z_d P_{d,d} &= 0, \\ P_{d-1,d-1} - \theta_{d-1} P_{d,d-1} - P_{d,d} &= 0, \\ x_d P_{d-1,d} - \varphi_d P_{d,d-1} - \theta_d^* P_{d,d} &= 0, \\ x_{d-1} P_{d-2,d} - \varphi_d P_{d-1,d-1} - \theta_d^* P_{d-1,d} + y_d z_d P_{d,d} &= 0. \end{aligned}$$

In these equations, eliminate $P_{d-1,d}$, $P_{d,d-1}$, $P_{d-1,d-1}$, $P_{d-2,d}$ to find that $P_{d,d}$ times the left-hand side of (74) is zero. By this and Lemma 17.2 we get (74). \square

18 Proof of Theorem 1.17(i)

Proof of Theorem 1.17(i). Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with fundamental parameter $\beta = 2$. By replacing A, A^* with their nonzero scalar multiples, we may assume that A, A^* has parameter array in Proposition 6.1. Note by Lemma 6.2 that $\text{Char}(\mathbb{F})$ is 0 or greater than d , and $s^2 \neq 1$. In view of Note 1.15, we may assume that the subdiagonal entries of A are all 1. We show that A, A^* is the pair (39) with $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.1. We use the Askey-Wilson relations for A, A^* . By Lemma 8.5 $\varrho = 4, \varrho^* = 4$, and $\omega = -2(s + s^{-1})$. Using (62) and (64) one finds

$$x_i = x_1 \quad (1 \leq i \leq d). \quad (75)$$

Using (63) and (65) one finds

$$y_i = y_1 \quad (1 \leq i \leq d). \quad (76)$$

By (66) and (67) for $i = 1$ together with (75) and (76),

$$x_1 + y_1 = s + s^{-1}. \quad (77)$$

By (66) for $i = 1$ together with (75)–(77),

$$(s + s^{-1} - 2x_1)(z_2 - 2z_1 + 2) = 0. \quad (78)$$

We claim that $s + s^{-1} - 2x_1 \neq 0$. By way of contradiction, assume $s + s^{-1} - 2x_1 = 0$, so $x_1 = (s + s^{-1})/2$. By this and (77), $y_1 = (s + s^{-1})/2$. Using these comments and (75), (76), we evaluate (68) to find that $(s - s^{-1})^2 = 0$, contradicting $s^2 \neq 1$. Thus the claim holds. By the claim and (78),

$$z_2 - 2z_1 + 2 = 0. \quad (79)$$

In (68) for $i = 1$, eliminate y_1 using (77), and eliminate z_2 using (79),

$$(x_1 - s)(x_1 - s^{-1}) = 0. \quad (80)$$

Thus either $x_1 = s$ or $x_1 = s^{-1}$. By replacing s with s^{-1} if necessary, we may assume $x_1 = s$. By this and (75) $x_i = s$ for $1 \leq i \leq d$. By $x_1 = s$ and (77) $y_1 = s^{-1}$. By this and (75), (76),

$$x_i = s \quad (1 \leq i \leq d), \quad (81)$$

$$y_i = s^{-1} \quad (1 \leq i \leq d). \quad (82)$$

By (66) with (81), (82),

$$(s - s^{-1})(z_{i-1} - 2z_i + z_{i+1} - 2) = 0 \quad (2 \leq i \leq d-1).$$

By this and $s^2 \neq 1$,

$$z_{i-1} - 2z_i + z_{i+1} - 2 = 0 \quad (2 \leq i \leq d-1).$$

Solve this recursion with (79) to find

$$z_i = i(z_1 - i + 1) \quad (1 \leq i \leq d).$$

So it suffices to show $z_1 = d$. Using (75), (76), we simplify (73) to find

$$s^{-2}(s - s^{-1})(z_1 - d) = 0.$$

This forces $z_1 = d$. The result follows. \square

19 Proof of Theorem 1.17(ii)

Proof of Theorem 1.17(ii). Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with fundamental parameter $\beta = -2$. By replacing A, A^* with their nonzero scalar multiples, we may assume that A, A^* has parameter array in Proposition 6.3. Note by Lemma 6.4 that $\text{Char}(\mathbb{F})$ is 0 or greater than d , and τ is not among $1 - d, 3 - d, \dots, d - 1$. By the assumption of Theorem 1.17 $d + 1$ does not vanish in \mathbb{F} . We show that A, A^* is the pair (39) with $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ as in Proposition 7.1. We use the Askey-Wilson relations for A, A^* . By Lemma 8.6 $\varrho = 4, \varrho^* = 4$, and $\omega = 4(d + 1)\tau$. By (62) and (64),

$$x_i = (-1)^{i-1}x_1 \quad (1 \leq i \leq d). \quad (83)$$

By (63) and (65),

$$y_i = (-1)^{i-1}y_1 \quad (1 \leq i \leq d). \quad (84)$$

By (66), (67) for $i = 1$ together with (83), (84),

$$y_1 = x_1 \quad (85)$$

By (66) for $1 \leq i \leq d - 1$, and using (83)–(85),

$$z_i = \begin{cases} iz_1 - i((d + 1)\tau x_1^{-1} + i - 1) & \text{if } i \text{ is even,} \\ iz_1 - (i - 1)((d + 1)\tau x_1^{-1} + i) & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d). \quad (86)$$

By (74)

$$2x_1(d + d\tau x_1 - x_1^2 z_1) = 0.$$

So

$$z_1 = d(1 + \tau x_1)x_1^{-2}.$$

By this and (86)

$$z_i = \begin{cases} i(dx_1^{-1} - i + 1) - ix_1^{-1}\tau & \text{if } i \text{ is even,} \\ i(dx_1^{-1} - i + 1) + (d - i + 1)x_1^{-1}\tau & \text{if } i \text{ is odd.} \end{cases} \quad (87)$$

By these comments and (74),

$$2d(x_1 - 1)(x_1 + 1) = 0.$$

So either $x_1 = 1$ or $x_1 = -1$. Setting $\epsilon = x_1$ we find that $\{x_i\}_{i=1}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d$ are as in Proposition 7.2. The result follows. \square

20 Proof of Theorem 1.17(iii)

In this section we prove Theorem 1.17(iii). Let A, A^* be a zero-diagonal TD-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Let β be the fundamental parameter of A, A^* , and assume $\beta \neq 2$, $\beta \neq -2$. By replacing A, A^* with their nonzero scalar multiples, we may assume that A, A^* has parameter array in Proposition 6.5 for nonzero $q, s \in \mathbb{F}$. We assume q is not a root of unity. Note by Lemma 3.4 that $\text{Char}(\mathbb{F}) \neq 2$. By Lemma 6.7

$$s^2 \neq q^i \quad (0 \leq i \leq d-1). \quad (88)$$

We use the Askey-Wilson relations for A, A^* . By Lemma 8.7

$$\begin{aligned} \varrho &= q^{d-2}(q^2 - 1)^2, & \varrho^* &= q^{d-2}(q^2 - 1)^2, \\ \omega &= -q^{-1}(q - 1)^2(q^{d+1} + 1)\tau, \end{aligned}$$

where

$$\tau = s + s^{-1}q^{d-1}.$$

Lemma 20.1 *The following hold:*

- (i) *Either $x_i x_{i-1}^{-1} = q$ ($1 \leq i \leq d$) or $x_i x_{i-1}^{-1} = q^{-1}$ ($1 \leq i \leq d$).*
- (ii) *Either $y_i y_{i-1}^{-1} = q$ ($1 \leq i \leq d$) or $y_i y_{i-1}^{-1} = q^{-1}$ ($1 \leq i \leq d$).*

Proof. (i): By (62) for $i = 2$,

$$x_1 - (q + q^{-1})x_2 + x_3 = 0.$$

By (64) for $i = 2$

$$x_1 x_2 - (q + q^{-1})x_1 x_3 + x_2 x_3 = 0.$$

In the above two equations, eliminate x_3 to find

$$(q + q^{-1})(x_1 - x_2 q)(x_1 - x_2 q^{-1}) = 0.$$

We have $q + q^{-1} \neq 0$ since q is not a root of unity. Therefore either $x_2 x_1^{-1} = q$ or $x_2 x_1^{-1} = q^{-1}$. Now solve the recursion (62) to get the result.

(ii) Similar. □

By Lemma 20.1 we have four cases:

Case 1: $x_i x_{i-1}^{-1} = q^{-1}$ and $y_i y_{i-1}^{-1} = q^{-1}$ for $1 \leq i \leq d$.

Case 2: $x_i x_{i-1}^{-1} = q^{-1}$ and $y_i y_{i-1}^{-1} = q$ for $1 \leq i \leq d$.

Case 3: $x_i x_{i-1}^{-1} = q$ and $y_i y_{i-1}^{-1} = q^{-1}$ for $1 \leq i \leq d$.

Case 4: $x_i x_{i-1}^{-1} = q$ and $y_i y_{i-1}^{-1} = q$ for $1 \leq i \leq d$.

Observe that Case 3 is reduced to Case 2 by replacing A, A^* with its anti-diagonal transpose. Similarly Case 4 is reduced to Case 1. So we consider Case 1 and Case 2.

20.1 Case 1

In this subsection we consider Case 1. So

$$x_i = x_1 q^{1-i}, \quad y_i = y_1 q^{1-i} \quad (1 \leq i \leq d). \quad (89)$$

Lemma 20.2 *We have*

$$(qx_1 - y_1)z_1 = (q-1)(q^d - 1)(q^{d-1}(q+1)y_1^{-1} - \tau), \quad (90)$$

$$(qx_1 - y_1)z_d = q^{d-1}(q-1)(q^d - 1)((q+1)x_1 - \tau). \quad (91)$$

Proof. Follows from (73) and (74). \square

Lemma 20.3 *For $1 \leq i \leq d$*

$$(x_1 - y_1 q)z_{i+1} - q^2(2x_1 - (q + q^{-1})y_1)z_i + q^4(x_1 - y_1 q^{-1})z_{i-1} + x_1 q^d(q^2 - 1)^2 - \tau q^i(q-1)^2(q^{d+1} + 1) = 0, \quad (92)$$

$$(y_1 - x_1 q)z_{i+1} - q^2(2y_1 - (q + q^{-1})x_1)z_i + q^4(y_1 - x_1 q^{-1})z_{i-1} + y_1 q^d(q^2 - 1)^2 - \tau q^i(q-1)^2(q^{d+1} + 1) = 0, \quad (93)$$

$$(x_1 - y_1 q)z_{i+1} + q(2y_1 - (q + q^{-1})x_1)z_i + q^2(x_1 - y_1 q^{-1})z_{i-1} - x_1^{-1} q^{d+2i-3}(q^2 - 1)^2 + \tau q^{i-1}(q-1)^2(q^{d+1} + 1) = 0. \quad (94)$$

Proof. Follows from (66)–(68). \square

Lemma 20.4 *Assume $x_1 \neq y_1$. Then for $1 \leq i \leq d$*

$$(x_1 + y_1)(z_i - q^2 z_{i-1} - q^d(q^2 - 1)) + \tau q^{i-1}(q-1)(q^{d+1} + 1) = 0. \quad (95)$$

Proof. In (92) and (93), eliminate z_{i+1} to find that $x_1 - y_1$ times (95) is 0. \square

First consider the case $x_1 \neq y_1$, $x_1 \neq -y_1$.

Lemma 20.5 *Assume $x_1 \neq y_1$ and $x_1 \neq -y_1$. Then $\tau = x_1 + y_1$, $x_1 y_1 = q^{d-1}$, and for $1 \leq i \leq d$*

$$z_i = q^{i-1}(q^i - 1)(q^{d-i+1} - 1). \quad (96)$$

Proof. By (95), for $1 \leq i \leq d$

$$z_i - q^2 z_{i-1} - q^d(q^2 - 1) + \frac{\tau q^{i-1}(q-1)(q^{d+1} + 1)}{x_1 + y_1} = 0.$$

Solve this recursion with $z_0 = 0$ to find that for $1 \leq i \leq d$,

$$z_i = q^d(q^{2i} - 1) - \frac{\tau q^{i-1}(q^i - 1)(q^{d+1} + 1)}{x_1 + y_1}. \quad (97)$$

By this and (91),

$$(q^d - 1)(x_1 + y_1 - \tau)((q^{d+2} + 1)x_1 - (q^{d+1} + q)y_1) = 0.$$

So either $\tau = x_1 + y_1$ or $(q^{d+1} + 1)x_1 = (q^{d+1} + q)y_1$.

First assume $\tau = x_1 + y_1$. By (97) with $\tau = x_1 + y_1$ we get (96). By (90),

$$(q^2 - 1)(q^d - 1)(q^{d-1} - x_1 y_1)y_1 = 0.$$

So $x_1 y_1 = q^{d-1}$.

Next assume $\tau \neq x_1 + y_1$. Then

$$y_1(q^{d+1} + q) = x_1(q^{d+2} + 1).$$

We have $q^{d+2} + 1 \neq 0$; otherwise both $q^{d+2} + 1 = 0$ and $q^{d+1} + q = 0$, so $q^2 = 1$, contradicting Lemma 6.7. Similarly, $q^d + 1 \neq 0$. Also note that $q^{d+1} + 1 \neq 0$; otherwise $y_1(q+1) = x_1(q+1)$ and so $x_1 = y_1$, contradicting the assumption. Now we find

$$y_1 = \frac{(q^{d+2} + 1)x_1}{q(q^d + 1)}.$$

By (94),

$$\tau = \frac{q^{d-1}(x_1 + y_1)(x_1 y_1 q^2 + 1)}{x_1 y_1 (q^{d+1} + 1)}.$$

By (90) with the above comments,

$$(q^{d+2} + 1)x_1^2 = q^d(q^d + 1).$$

Using these comments we find $\tau = x_1 + y_1$, a contradiction. \square

Next consider the case $x_1 = -y_1$.

Lemma 20.6 *Assume $x_1 = -y_1$. Then $\tau = 0$, $x_1^2 = -q^{d-1}$, and for $1 \leq i \leq d$*

$$z_i = q^{i-1}(q^i - 1)(q^{d-i+1} - 1). \quad (98)$$

Proof. First assume $q^{d+1} \neq -1$. Then $\tau = 0$ by (95). In (92) and (94), eliminate z_{i+1} to find that for $1 \leq i \leq d$

$$z_i - qz_{i-1} - q^d(q-1)(1 + x_1^{-2}q^{2i-3}) = 0.$$

Solving this recursion with $z_0 = 0$,

$$z_i = q^d(q^i - 1)(1 + x_1^{-2}q^{i-2}). \quad (99)$$

By (91),

$$(q+1)(q^d - 1)(x_1^2 + q^{d-1}) = 0.$$

So $x_1^2 = -q^{d-1}$. Now (98) follows from (99).

Next assume $q^{d+1} = -1$. By (90)

$$z_1 = \frac{(q-1)(q^2x_1\tau - q - 1)}{q^3x_1^2}.$$

By this and (92) for $i = 1$,

$$z_2 = \frac{(q^2 - 1)(q(q-1)x_1^2 + q^2\tau x_1 - q - 1)}{q^2x_1^2}.$$

Using these comments, evaluate (94) for $i = 1$ to find

$$(q^2 - 1)^2(q^2x_1^2 + q^2\tau x_1 - 1) = 0.$$

By this

$$\tau = -x_1 + x_1^{-1}q^{-2}.$$

Using these comments, solve the recursion (92) to find that

$$z_i = -q^{-1}(q^i - 1)(1 + x_1^{-2}q^{i-2}) \quad (1 \leq i \leq d).$$

By this and (94) for $i = d$,

$$2(q+1)(x_1^2 - q^{-2}) = 0.$$

So $x_1^2 = q^{-2}$. By these comments, $\tau = 0$ and

$$z_i = -q^{-1}(q^i - 1)(1 + q^i) \quad (1 \leq i \leq d).$$

Now the result follows. \square

Next consider the case $x_1 = y_1$.

Lemma 20.7 *Assume $x_1 = y_1$. Then d is even, $\tau = x_1 + x_1^{-1}q^{d-1}$, and for $1 \leq i \leq d$*

$$z_i = \begin{cases} q^d(q^i - 1)(1 - x_1^{-2}q^{i-2}) & \text{if } i \text{ is even,} \\ -q^{i-1}(q^{d-i+1} - 1)(1 - x_1^{-2}q^{d+i-1}) & \text{if } i \text{ is odd.} \end{cases} \quad (100)$$

Proof. In (92) and (94), eliminate z_{i+1} to find

$$z_i + qz_{i-1} + q^d(q+1)(1 + x_1^{-2}q^{2i-3}) - \tau x_1^{-1}q^{i-1}(q^{d+1} + 1) = 0.$$

Solving this recursion with $z_0 = 0$, we get

$$z_i = \begin{cases} q^d(q^i - 1)(1 - x_1^{-2}q^{i-2}) & \text{if } i \text{ is even,} \\ -q^d(q^i + 1)(1 + x_1^{-2}q^{i-2}) + \tau x_1^{-1}q^{i-1}(q^{d+1} + 1) & \text{if } i \text{ is odd.} \end{cases} \quad (101)$$

By this and (90),

$$q^{d-1}(q^2 - 1)(x_1 + x_1^{-1}q^{d-1} - \tau) = 0.$$

So $\tau = x_1 + x_1^{-1}q^{d-1}$. By this and (101) we get (100). We show d is even. By way of contradiction, assume d is odd. By (91)

$$z_d = q^d(q^d - 1)(1 - x_1^{-2}q^{d-2}).$$

By (100)

$$z_d = -q^{d-1}(q - 1)(1 - x_1^{-2}q^{2d-1}).$$

Comparing these two equations,

$$(q^{d+1} - 1)(x_1^2 - q^{d-1}) = 0.$$

So either $q^{d+1} = 1$ or $x_1^2 = q^{d-1}$. First assume $q^{d+1} = 1$. Set $r = (d + 1)/2$, and observe r is an integer such that $2 \leq i \leq d - 1$. We have $q^r = \pm 1$, contradicting Lemma 6.7(i), (ii). Next assume $x_1^2 = q^{d-1}$. By $\tau = x_1 + x_1^{-1}q^{d-1}$ and $\tau = s + s^{-1}q^{d-1}$ we have either $x_1 = s$ or $x_1 = s^{-1}q^{d-1}$. In either case $s^2 = q^{d-1}$, contradicting Lemma 6.7(iii). \square

Lemma 20.8 *At least one of the following holds:*

(i) $x_1y_1 = q^{d-1}$, $\tau = x_1 + x_1^{-1}q^{d-1}$, and for $1 \leq i \leq d$

$$z_i = q^{i-1}(q^i - 1)(q^{d-i+1} - 1).$$

(ii) $x_1 = y_1$, $\tau = x_1 + x_1^{-1}q^{d-1}$, and for $1 \leq i \leq d$

$$z_i = \begin{cases} q^d(q^i - 1)(1 - x_1^{-2}q^{i-2}) & \text{if } i \text{ is even,} \\ -q^{i-1}(q^{d-i+1} - 1)(1 - x_1^{-2}q^{d+i-1}) & \text{if } i \text{ is odd.} \end{cases}$$

Proof. First assume $x_1 \neq y_1$ and $x_1 \neq -y_1$. Then case (i) occurs by Lemma 20.5. Next assume $x_1 = -y_1$. Then case (i) occurs by Lemma 20.6. Next assume $x_1 = y_1$. Then case (ii) occurs by Lemma 20.7. \square

20.2 Case 2

In this subsection we consider Case 2. So

$$x_i = x_1q^{1-i}, \quad y_i = y_1q^{i-1} \quad (1 \leq i \leq d). \quad (102)$$

Lemma 20.9 *We have*

$$(x_1 - qy_1)z_1 = (q - 1)(q^d - 1)((q + 1)y_1^{-1} - q\tau), \quad (103)$$

$$(x_1 - q^{2d-3}y_1)z_d = q^{d-2}(q - 1)(q^d - 1)((q + 1)x_1 - \tau). \quad (104)$$

Proof. Follows from (73) and (74). \square

Lemma 20.10 For $1 \leq i \leq d$

$$(x_1 - y_1 q^{2i+1})z_{i+1} - q^2(2x_1 - y_1 q^{2i-2}(q + q^{-1}))z_i + q^4(x_1 - y_1 q^{2i-5})z_{i-1} + x_1 q^d (q^2 - 1)^2 - \tau q^i (q - 1)^2 (q^{d+1} + 1) = 0, \quad (105)$$

$$(x_1 - y_1 q^{2i+1})z_{i+1} - q(x_1(q + q^{-1}) - 2y_1 q^{2i-2})z_i + q^2(x_1 - y_1 q^{2i-5})z_{i-1} - y_1 q^{d+2i-3}(q^2 - 1)^2 + \tau q^{i-1}(q - 1)^2(q^{d+1} + 1) = 0, \quad (106)$$

$$(x_1 - y_1 q^{2i+1})z_{i+1} - q(x_1(q + q^{-1}) - 2y_1 q^{2i-2})z_i + q^2(x_1 - y_1 q^{2i-5})z_{i-1} - x_1^{-1} q^{d+2i-3}(q^2 - 1)^2 + \tau q^{i-1}(q - 1)^2(q^{d+1} + 1) = 0. \quad (107)$$

Proof. Follows from (66)–(68). \square

Lemma 20.11 We have $x_1 y_1 = 1$.

Proof. Compare (106) and (107). \square

Lemma 20.12 For $1 \leq i \leq d$

$$(x_1^2 - q^{2i-1})z_i - q^2(x_1^2 - q^{2i-5})z_{i-1} - q^d(x_1^2 + q^{2i-3})(q^2 - 1) + \tau x_1 q^{i-1}(q - 1)(q^{d+1} + 1) = 0. \quad (108)$$

Proof. In (105) and (106), eliminate z_{i+1} and simplify the result using $y_1 = x_1^{-1}$. \square

Lemma 20.13 For $1 \leq i \leq d$

$$(x_1^2 - q^{2i-3})(x_1^2 - q^{2i-1})z_i = (q^i - 1)(x_1^2 - q^{i-2})(q^d(q^i + 1)(x_1^2 + q^{i-2}) - \tau x_1 q^{i-1}(q^{d+1} + 1)). \quad (109)$$

Proof. Solve the recursion (108) with $z_0 = 0$. \square

Lemma 20.14 We have

$$\tau = x_1 + x_1^{-1} q^{d-1}. \quad (110)$$

Proof. In (103) and (108) for $i = 1$, eliminate z_1 to find

$$(q^2 - 1)^2(x_1 + x_1^{-1} q^{d-1} - \tau) = 0.$$

So (110) holds. \square

Lemma 20.15 For $1 \leq i \leq d$

$$(x_1^2 - q^{2i-1})z_i - q^2(x_1^2 - q^{2i-5})z_{i-1} - q^d(q^2 - 1)(x_1^2 + q^{2i-3}) + q^{i-1}(q - 1)(q^{d+1} + 1)(x_1^2 + q^{d-1}) = 0. \quad (111)$$

Proof. Follows from (108) and (110). \square

Lemma 20.16 For $1 \leq i \leq d$

$$(x_1^2 - q^{2i-3})(x_1^2 - q^{2i-1})z_i = q^{i-1}(q^i - 1)(q^{d-i+1} - 1)(x_1^2 - q^{i-2})(x_1^2 - q^{d+i-1}). \quad (112)$$

Proof. Follows from (109) and (110). \square

Lemma 20.17 We have $x_1^2 \neq q^{2i-1}$ for $1 \leq i \leq d - 1$, and

$$z_1 = \frac{(q - 1)(q^d - 1)(x_1^2 - q^d)}{x_1^2 - q}, \quad (113)$$

$$z_i = \frac{q^{i-1}(q^i - 1)(q^{d-i+1} - 1)(x_1^2 - q^{i-2})(x_1^2 - q^{d+i-1})}{(x_1^2 - q^{2i-3})(x_1^2 - q^{2i-1})} \quad (2 \leq i \leq d - 1). \quad (114)$$

Proof. By (103)

$$(x_1^2 - q)z_1 = (q - 1)(q^d - 1)(x_1^2 - q^d).$$

We have $x_1^2 \neq q$; otherwise $0 = (q - 1)(q^d - 1)(q - q^d)$, contradicting Lemma 6.7(i). So (113) holds. We claim that $x_1^2 \neq q^{2i-1}$ for $2 \leq i \leq d - 1$. By way of contradiction, assume $x_1^2 = q^{2i-1}$ for some i ($2 \leq i \leq d - 1$). In (112), the left-hand side vanishes, so

$$0 = q^{i-1}(q^i - 1)(q^{d-i+1} - 1)(q^{2i-1} - q^{i-2})(q^{2i-1} - q^{d+i-1}).$$

This is a contradiction by Lemma 6.7(i). So the claim holds. Now (114) follows from (112). \square

Lemma 20.18 Assume $x_1^2 = q^{2d-1}$. Then

$$z_i = \frac{q^{d-i}(q^i - 1)(q^{2d-i+1} - 1)}{(q^{d-i} + 1)(q^{d-i+1} + 1)} \quad (1 \leq i \leq d - 1), \quad (115)$$

$$z_d = \frac{(q^d - 1)(q^{d+1} - 1)}{q + 1}. \quad (116)$$

Proof. Line (115) follows from Lemma 20.17. Line (116) follows from (104). \square

Lemma 20.19 Assume $x_1^2 \neq q^{2d-1}$. Then

$$z_d = \frac{q^{d-1}(q-1)(q^d-1)(x_1^2 - q^{d-2})}{x_1^2 - q^{2d-3}}. \quad (117)$$

Proof. Follows from (104). \square

Lemma 20.20 We have $x_1 y_1 = 1$, and at least one of the following holds:

(i) $x_1^2 = q^{2d-1}$, $\tau = x_1 + x_1^{-1}q^{d-1}$, and

$$\begin{aligned} z_i &= \frac{q^{d-i}(q^i - 1)(q^{2d-i+1} - 1)}{(q^{d-i} + 1)(q^{d-i+1} + 1)} \quad (1 \leq i \leq d-1), \\ z_d &= \frac{(q^d - 1)(q^{d+1} - 1)}{q + 1}. \end{aligned}$$

(ii) $x_1^2 \neq q^{2i-1}$ for $1 \leq i \leq d$, $\tau = x_1 + x_1^{-1}q^{d-1}$, and

$$\begin{aligned} z_1 &= \frac{(q-1)(q^d-1)(x_1^2 - q^d)}{x_1^2 - q}, \\ z_i &= \frac{q^{i-1}(q^i - 1)(q^{d-i+1} - 1)(x_1^2 - q^{i-2})(x_1^2 - q^{d+i-1})}{(x_1^2 - q^{2i-3})(x_1^2 - q^{2i-1})} \quad (2 \leq i \leq d-1), \\ z_d &= \frac{q^{d-1}(q-1)(q^d-1)(x_1^2 - q^{d-2})}{x_1^2 - q^{2d-3}}. \end{aligned}$$

Proof. First assume $x_1^2 = q^{2d-1}$. Then case (i) occurs by Lemma 20.18. Next assume $x_1^2 \neq q^{2d-1}$. Then case (ii) occurs by Lemmas 20.17 and 20.19. \square

20.3 Completing the proof of Theorem 1.17(iii)

By Lemmas 20.8 and 20.20, we have one of cases (i), (ii) in Lemma 20.8 and cases (i), (ii) in Lemma 20.20. In either case we have $\tau = x_1 + x_1^{-1}q^{d-1}$. By this and $\tau = s + s^{-1}q^{d-1}$, we have either $x_1 = s$ or $x_1 = s^{-1}q^{d-1}$. In view of Note 6.6, we may assume $x_1 = s$ by replacing s with $s^{-1}q^{d-1}$ if necessary. First assume (i) in Lemma 20.8 occurs. Then $\{x_i\}_{i=1}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ are as in Proposition 7.3. Next assume case (ii) in Lemma 20.8 occurs. Then $\{x_i\}_{i=1}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ are as in Proposition 7.5. Next assume case (i) in Lemma 20.20 occurs. Then $\{x_i\}_{i=1}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ are as in Proposition 7.4 with $s^2 = q^{2d-1}$. Next assume case (ii) in Lemma 20.20 occurs. We have $s^2 \neq q^{2i-1}$ for $1 \leq i \leq d$. By Lemma 6.7(iii) $s^2 \neq q^{2i}$ for $0 \leq i \leq d-1$. So $s^2 \neq q^i$ for $0 \leq i \leq 2d-1$. Now $\{x_i\}_{i=1}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ are as in Proposition 7.4. This completes the proof of Theorem 1.17(iii).

21 Acknowledgement

The author thanks Paul Terwilliger for many insightful comments that lead to great improvements in the paper.

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Keywords. Leonard pair, tridiagonal pair, Askey-Wilson relation, orthogonal polynomial

2010 Mathematics Subject Classification. 05E35, 05E30, 33C45, 33D45